

# On Tate Acyclicity and Uniformity of Berkovich Spectra and Adic Spectra

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## Abstract

We construct an example of a uniform Banach algebra such that a rational localisation of the Berkovich spectrum does not preserve the uniformity. It is an example of a non-sheafy uniform Banach algebra. We also construct examples of uniform affinoid rings such that rational localisations of the adic spectra do not preserve the uniformity. One of them is an example of a non-sheafy uniform affinoid ring.

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## 0 Introduction

An affinoid ring  $(\mathcal{A}^\flat, \mathcal{A}^+)$  is said to be *sheafy* if the structure presheaf on the topology of the adic spectrum  $\mathrm{Spa}(\mathcal{A}^\flat, \mathcal{A}^+)$  is a sheaf. Similarly, a Banach  $k$ -algebra  $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$  is said to be *sheafy* if the Banach algebra of sections on a rational domain of the Berkovich spectrum is independent of the choice of its presentation, and if the structure presheaf on the Grothendieck topology generated by rational domains is a sheaf. We note that the Banach  $k$ -algebra of sections associated to a rational domain of a Berkovich spectrum deeply depends on the choice of its presentation in general, because the universality of an affinoid domain does not necessarily hold. This is one of the essential differences between a Berkovich spectrum and an adic spectrum. Even if two rational domains in an adic spectrum share all points of height 1, they do not coincide in general. We construct an example of a uniform Banach algebra such that a rational localisation of the Berkovich spectrum does not preserve the uniformity. It is an example of a non-sheafy uniform Banach algebra. We also construct examples of uniform affinoid rings such that rational localisations of the adic spectra do not preserve the uniformity. One of them is an example of a non-sheafy uniform affinoid ring. We explain briefly the motivation of constructing such examples.

First of all, we are interested in whether there is a good criterion of the sheaf condition. We recall several known facts for it.

**Theorem 0.1** (J. Tate). *Every affinoid  $k$ -algebra is sheafy.*

**Theorem 0.2** (V. G. Berkovich). *Every  $k$ -affinoid algebra is sheafy.*

**Theorem 0.3** (R. Huber). *Every strongly Noetherian  $f$ -adic Tate ring is sheafy.*

**Theorem 0.4** (P. Scholze). *Every perfectoid affinoid  $K$ -algebra is sheafy.*

Theorem 0.1 and Theorem 0.2 are results for Berkovich spectra, and Theorem 0.3 and Theorem 0.4 are results for adic spectra. Here the base field  $k$  in Theorem 0.1 and Theorem 0.2 is a complete valuation field of height at most 1, and the base field  $K$  in Theorem 0.4 is a perfectoid field. The great appearance of perfectoid theory yields an expectation that there is a good wide isomorphism class of Banach algebras (resp. Banach affinoid rings) containing both reduced affinoid  $k$ -algebras and perfectoid  $K$ -algebras (resp. reduced strongly Noetherian  $f$ -adic Tate rings and perfectoid affinoid  $K$ -algebras). For example, consider the class of uniform Banach  $k$ -algebras (resp. uniform affinoid rings).

**Question 0.5.** *Is every uniform Banach  $k$ -algebra (resp. every uniform affinoid ring) sheafy?*

We remark that the notion of the uniformity of an affinoid ring differs from that of a Banach  $k$ -algebra. The former one corresponds to the notion of a Banach function algebra, because an affinoid ring whose topology is given by a norm naturally forgets the

norm. We also consider the property of an affinoid ring precisely corresponding to the uniformity of a Banach  $k$ -algebra. We call it the strong uniformity.

The sheaf condition for an affinoid  $k$ -algebra (resp. a strongly Noetherian  $f$ -adic Tate ring) is proved by reducing it to the simplest case where the rational covering is given by the Weierstrass domain and the Laurent domain associated to a common single element. For a general uniform Banach  $k$ -algebra (resp. uniform affinoid ring), the sheaf condition for such a covering is easily verified, but it can not be reduced to such a case. It is because there is no known good condition which is stronger than the uniformity and which is preserved under a rational localisation. In particular, a natural question arises:

**Question 0.6.** *Is the uniformity preserved under a rational localisation?*

These two questions are not ignorable for us to construct a new comprehensive theory in rigid geometry, and would be considered many times by many people before the birth of perfectoid theory. Now they reappeared in Scholze's talk in "Hot Topics: Perfectoid Spaces and their Applications" held in MSRI in February 2014, and we recognised the importance of these questions again. The answer of Question 0.6 is NO both for a Berkovich spectrum and for an adic spectrum as we stated at the beginning. We will give counter-examples in Theorem 2.4 for a Berkovich spectrum, and Theorem 3.11 and Theorem 3.13 for adic spectra. Furthermore, we verify in Corollary 2.5 that the counter-example in Theorem 2.4 is not sheafy. Therefore the answer of Question 0.5 is NO for a Berkovich spectrum. We remark that the result of Theorem 3.11 obviously includes the fact that the strong uniformity of an adic ring is not preserved under a rational localisation. It just means that at least the bounded component  $\mathcal{O}^\circ$  of the structure presheaf  $\mathcal{O}$  is not a sheaf for the adic spectrum of a general strongly uniform affinoid ring. This corresponds to a computation of the Čech cohomology  $\check{H}^1(\mathcal{O}^\circ)$ .

We do not have an answer to Question 0.5 for a uniform affinoid ring yet. Though a rational localisation does not preserve the uniformity, it is still significant to compute  $\check{H}^1(\mathcal{O})$  for rational coverings of the total space. It is easily reduced to the case that an affinoid ring is obtained as the uniform Banach  $k$ -algebra corresponding to a closed subset  $\Sigma$  of the Berkovich spectrum of a uniform Banach  $k$ -algebra topologically of finite type. However, in the process of computing the submetric direct limit of admissible exact sequences associated to neighbourhoods of  $\Sigma$  obtained as unions of affinoid domains, we have to determine the annihilator of  $\check{H}^2(\mathcal{O}^\circ)$  on the Berkovich spectra of uniform Banach  $k$ -algebras topologically of finite type. This obstruction  $\check{H}^2(\mathcal{O}^\circ)$  is much more difficult to calculate than  $\check{H}^1(\mathcal{O}^\circ)$ . We verify in Theorem 3.15 that the example in Theorem 3.13 is also an example of a non-sheafy uniform affinoid ring. Thus the answer of Question 0.5 is NO also for an adic spectrum.

# 1 Preliminaries

Throughout this paper, let  $k$  denote a complete valuation field whose valuation is non-trivial and height 1. Fixing an order preserving embedding of the value group of  $k$  into the multiplicative group  $(0, \infty)$ , we regard the valuation as a map  $|\cdot|: k \rightarrow [0, \infty)$ . A  $k$ -algebra is always assumed to be a commutative associative unital  $k$ -algebra. In this section, we recall definitions and basic properties of Banach spaces and Banach algebras.

## 1.1 Banach Spaces

A *seminorm* of a  $k$ -vector space  $V$  is a map  $\|\cdot\|_V: V \rightarrow [0, \infty)$  satisfying the following conditions:

- (i)  $\|v - v'\|_V \leq \max\{\|v\|_V, \|v'\|_V\}$  for any  $(v, v') \in V \times V$ .
- (ii)  $\|cv\|_V = |c| \|v\|_V$  for any  $(c, v) \in k \times V$ .

A *seminormed  $k$ -vector space* is a pair  $(V, \|\cdot\|_V)$  of a  $k$ -vector space  $V$  and a seminorm  $\|\cdot\|_V$ . We endow  $V$  with the pseudo-metric  $V \times V \rightarrow [0, \infty): (v, v') \mapsto \|v - v'\|_V$  associated to  $\|\cdot\|_V$ . We call  $V$  the underlying  $k$ -vector space of  $(V, \|\cdot\|_V)$ . We put  $\|\cdot\|_{(V, \|\cdot\|_V)} := \|\cdot\|_V$ , and call it the seminorm of  $(V, \|\cdot\|_V)$ .  $V$  also denotes  $(V, \|\cdot\|_V)$ , and hence  $\|\cdot\|_V$  is called the seminorm of  $V$ .

Let  $V$  and  $W$  be seminormed  $k$ -vector spaces. A  $k$ -linear homomorphism  $\varphi: V \rightarrow W$  is said to be *bounded* if there is a  $C > 0$  such that  $\|\varphi(v)\|_W \leq C\|v\|_V$  for any  $v \in V$ , is said to be *submetric* if  $\|\varphi(v)\|_W \leq \|v\|_V$  for any  $v \in V$ , and is said to be *isometric* if  $\|\varphi(v)\|_W = \|v\|_V$  for any  $v \in V$ . We note that the boundedness is equivalent to the continuity with respect to the topologies given by the pseudo-metrics under the assumption that the valuation of  $k$  is non-trivial. A bounded  $k$ -linear homomorphism  $\varphi: V \rightarrow W$  is said to be *admissible* if the inverse of the induced bijective homomorphism  $V/\ker(\varphi) \rightarrow \text{im}(\varphi)$  is bounded with respect to the quotient seminorm on  $V/\ker(\varphi)$  and the restriction of  $\|\cdot\|_W$  on  $\text{im}(\varphi)$ . Namely, the admissibility is equivalent to the condition  $\text{coker}(\varphi) \cong \text{im}(\varphi)$  in the category of seminormed  $k$ -vector spaces and bounded  $k$ -linear homomorphisms.

A *norm* of a  $k$ -vector space  $V$  is a seminorm  $\|\cdot\|_V$  satisfying the following condition:

- (iii)  $\|v\|_V \neq 0$  for any  $v \in V \setminus \{0\}$ .

A *normed  $k$ -vector space* is a seminormed  $k$ -vector space whose seminorm is a norm. The pseudo-metric associated to  $\|\cdot\|_V$  is a metric, and hence we call it the metric associated to  $\|\cdot\|_V$ . A norm  $\|\cdot\|_V$  of  $V$  is said to be *complete* if the metric associated to  $\|\cdot\|_V$  is complete. A *Banach  $k$ -vector space* is a normed  $k$ -vector space whose norm is complete.

Let  $V$  be a seminormed  $k$ -vector space. For each  $r \in (0, \infty)$ , we set

$$\begin{aligned} V(r) &:= \{v \in V \mid \|v\|_V \leq r\} \\ V(r-) &:= \{v \in V \mid \|v\|_V < r\}. \end{aligned}$$

They are clopen subsets. We have  $k(r)V(1) \subset V(r)$  and  $k(\rho)V(r) = V(\rho r)$  for any  $r \in (0, \infty)$  and  $\rho \in |k^\times|$ . Since the valuation of  $k$  is assumed to be non-trivial, the multiplication  $k \otimes_{k(1)} V(1) \rightarrow V$  is an isomorphism of  $k$ -vector spaces. If  $V$  is a Banach  $k$ -vector space, the completeness guarantees

$$V(1) \cong \varprojlim_{0 < r < 1} V(1)/V(r) \cong \varprojlim_{0 < r < 1} V(1)/k(r)V(1)$$

as topological  $k(1)$ -modules. Therefore the topology (and hence the equivalent class of the norm) of  $V$  is determined by the algebraic structure of the integral model  $V(1)$ , while the norm of  $V$  can not unless the closure of  $|k| \subset [0, \infty)$  contains  $\|V\|$ . For more detail, see the last assertion of Proposition 1.5.

**Definition 1.1.** Let  $V$  be a  $k$ -vector space. For a  $k(1)$ -submodule  $W \subset V$ , we set

$$W^{\text{ac}} := \bigcap_{r \in (1, \infty)} k(r)W.$$

We say that  $W$  is *adically closed* if  $W = W^{\text{ac}}$ . We say that  $W$  is a *lattice* (of  $V$ ) if the multiplication  $k \otimes_{k(1)} W \rightarrow V$  is an isomorphism of  $k$ -vector spaces.

We have  $W^{\text{acac}} = W^{\text{ac}}$ , and hence  $W^{\text{ac}}$  is the smallest adically closed  $k(1)$ -submodule of  $V$  containing  $W$ . If the valuation of  $k$  is discrete, then  $W^{\text{ac}} = W$ . On the other hand, if  $|k|$  is dense in  $[0, \infty)$ , then the multiplication  $k(1-) \otimes_{k(1)} k(1-) \rightarrow k(1-)$  is an isomorphism, and  $W^{\text{ac}}$  coincides with the significant  $k(1)$ -module

$$W_*^{\text{a}} := \left\{ v \in V \mid \epsilon v \in W, \forall \epsilon \in k(1-) \right\}$$

in almost mathematics introduced in [GR].

**Definition 1.2.** A  $k(1)$ -module  $M$  is said to be *adically separated* if  $\bigcap_{r \in (0, 1)} k(r)M = 0$ .

**Proposition 1.3.** Let  $V$  be a normed  $k$ -vector space. Then every bounded  $k(1)$ -submodule  $W \subset V$  is adically separated.

*Proof.* Take an  $R \in (0, \infty)$  with  $W \subset V(R)$ . Then we have

$$\bigcap_{r \in (0, 1)} k(r)W \subset \bigcap_{r \in (0, 1)} k(r)V(R) \subset \bigcap_{r \in (0, 1)} V(rR) = 0$$

because  $\|\cdot\|_V$  is a norm. Therefore  $W$  is adically separated.  $\square$

**Definition 1.4.** Let  $V$  be a  $k$ -vector space, and  $W \subset V$  a lattice. We call

$$\begin{aligned} \|\cdot\|_{V, W} : V &\rightarrow [0, \infty) \\ v &\mapsto \inf \{ r \in (0, \infty) \mid v \in k(r)W \} \end{aligned}$$

the *seminorm on  $V$  associated to  $W$* . If  $W$  is adically separated, then we call it the *norm on  $V$  associated to  $W$* .

The closed unit ball of  $V$  with respect to  $\|\cdot\|_{V,W}$  coincides with  $W^{\text{ac}}$  by definition. We remark that  $\|\cdot\|_{V,W^{\text{ac}}} = \|\cdot\|_{V,W}$  because  $W^{\text{acac}} = W^{\text{ac}}$ . The seminorm  $\|\cdot\|_{V,W}$  is a norm if and only if  $W$  is adically separated. Therefore  $W$  is adically separated if and only if so is  $W^{\text{ac}}$ .

**Proposition 1.5.** *For any seminormed  $k$ -vector space  $V$ ,  $V(1)$  is an adically closed lattice. If  $\|\cdot\|_V$  is a norm, then  $V(1)$  is adically separated. If the closure of  $|k| \subset [0, \infty)$  contains  $\|V\|_V$ , then  $\|\cdot\|_V$  coincides with  $\|\cdot\|_{V,V(1)}$ .*

The last assertion guarantees that the algebraic structure of the integral model  $V(1)$  determine the norm of  $V$  if  $|k|$  is dense in  $[0, \infty)$ . Therefore if a reader is not so accustomed to dealing with a norm, then it might be helpful to assume that  $k$  is highly ramified.

*Proof.* The multiplication  $k \otimes_{k(1)} V(1) \rightarrow V$  is an isomorphism because the valuation of  $k$  is non-trivial, and hence  $V(1)$  is a lattice. The equality  $V(1)^{\text{ac}} = V(1)$  holds automatically when the valuation of  $k$  is discrete. Therefore we may assume that  $|k|$  is dense in  $[0, \infty)$ . Let  $v \in V(1)^{\text{ac}} = V(1)_*^{\text{a}}$ . For any  $r \in (1, \infty)$ , there is an  $\epsilon \in k^\times$  such that  $r^{-1} < |\epsilon| < 1$ . Since  $v \in V(1)_*^{\text{a}}$ , we have  $\epsilon v \in V(1)$ . It guarantees  $\|v\|_V = |\epsilon|^{-1} \|\epsilon v\|_V < r$ . It implies  $\|v\|_V \leq 1$ , and hence  $v \in V(1)$ . Therefore  $V(1)^{\text{ac}} = V(1)$ . Thus  $V(1)$  is adically closed. If  $\|\cdot\|_V$  is a norm, then  $V(1)$  is adically separated by Proposition 1.3.

We verify the second assertion. Suppose that  $|k| \subset [0, \infty)$  contains  $\|V\|_V$ . Let  $v \in V$ . For any  $r \in (\|v\|_V, \infty)$ , take a  $c \in k^\times$  such that  $\|v\|_V < |c| \leq r$ , and then  $v = c(c^{-1}v) \in k(r)V(1)$ . Therefore  $\|v\|_{V,V(1)} \leq \|v\|_V$ . For any  $r \in (0, \|v\|_V)$ , we have  $k(r)V(1) \subset V(r) \subset V \setminus \{v\}$ . Therefore  $\|v\|_V \leq \|v\|_{V,V(1)}$ . Thus  $\|v\|_{V,V(1)} = \|v\|_V$ .  $\square$

In particular, if  $|k|$  is dense in  $[0, \infty)$ , then there is an anti-order preserving one-to-one correspondence between the set of adically closed lattices of  $V$  (resp. adically separated adically closed lattices of  $V$ ) and the set of seminorms on  $V$  (resp. norms on  $V$ ) by Proposition 1.5.

**Lemma 1.6.** *Let  $V$  be a seminormed  $k$ -vector space. Denote by  $\mathcal{V}$  the completion of  $V$  with respect to  $\|\cdot\|_V$ . Then for any  $k(1)$ -submodule  $W \subset V$  with  $W^{\text{ac}} = V(1)$ , the closure  $\mathcal{W} \subset \mathcal{V}$  of the image of  $W$  is an adically separated lattice with  $\mathcal{W}^{\text{ac}} = \mathcal{V}(1)$ . Moreover, the canonical homomorphism*

$$k \otimes_{k(1)} \varprojlim_{0 < r < 1} W/k(r)W \rightarrow \mathcal{V}$$

*is an isomorphism of  $k$ -vector spaces, and its restriction gives an isomorphism*

$$\varprojlim_{0 < r < 1} W/k(r)W \rightarrow \mathcal{W}$$

*of topological  $k(1)$ -modules with respect to the inverse limit topology.*

*Proof.* Let  $M \subset \mathcal{V}(1)$  be an arbitrary  $k(1)$ -submodule with  $M^{\text{ac}} = \mathcal{V}(1)$ . We have  $k(r)\mathcal{V}(1) \subset M$  for any  $r \in (0, 1)$ , and hence  $M$  is a lattice because the valuation of  $k$  is non-trivial. By Proposition 1.3,  $M$  is adically separated. Therefore in order to verify the first assertion, it suffices to show that  $\mathcal{W}^{\text{ac}} = \mathcal{V}(1)$ . Since  $W \subset W^{\text{ac}} = V(1)$ , we have  $\mathcal{W} \subset \mathcal{V}(1)$  because  $\|\cdot\|_{\mathcal{V}}$  is an extension of  $\|\cdot\|_V$ . Since  $\mathcal{V}(1)$  is adically closed by Proposition 1.5, we obtain  $\mathcal{W}^{\text{ac}} \subset \mathcal{V}(1)$ . Denote by  $\varphi: V \rightarrow \mathcal{V}$  the canonical bounded homomorphism. Take a  $v \in \mathcal{V}(1)$ . Take a sequence  $(v_i)_{i \in \mathbb{N}} \in V^{\mathbb{N}}$  such that  $\lim_{i \rightarrow \infty} \varphi(v_i) = v$ . Since  $\|v\|_{\mathcal{V}} \leq 1$ , there is an  $N \in \mathbb{N}$  such that  $v_i \in V(1)$  for any  $i \geq N$ . In particular,  $\mathcal{V}(1)$  is the closure of the image of  $V(1)$ . Therefore if the valuation of  $k$  is trivial, then we obtain  $\mathcal{W} = \mathcal{V}(1)$  because  $W = W^{\text{ac}} = V(1)$ . Suppose that  $|k|$  is dense in  $[0, \infty)$ . Let  $\epsilon \in k(1-)$ . Since  $W^{\text{ac}} = V(1)$ ,  $\epsilon v_i \in W$  for any  $i \geq N$ , and hence  $\epsilon v = \lim_{i \rightarrow \infty} \varphi(\epsilon v_i) \in \mathcal{W}$ . Thus  $v \in \mathcal{W}^{\text{ac}}$ . We conclude  $\mathcal{W}^{\text{ac}} = \mathcal{V}(1)$ .

The kernel of the canonical homomorphism  $W \rightarrow \varprojlim_{0 < r < 1} W/k(r)W$  coincides with  $\bigcap_{r \in (0,1)} k(r)W$ . For any  $r, r' \in (0, 1)$  with  $r' < r$ , we have  $k(r')V(1) \subset k(r)W \subset k(r)V(1)$  because  $W^{\text{ac}} = V$ . Therefore we obtain

$$\bigcap_{r \in (0,1)} k(r)W = \bigcap_{r \in (0,1)} k(r)V(1) = \{ v \in V \mid \|v\|_V = 0 \}$$

It implies that  $\varphi|_W: W \rightarrow \mathcal{W}$  extends to a unique continuous homomorphism

$$\iota: \varprojlim_{0 < r < 1} W/k(r)W \rightarrow \mathcal{W}$$

by the universality of an inverse limit of topological  $k(1)$ -modules, and is an isomorphism of topological  $k(1)$ -modules because it is the composite of natural isomorphisms  $\varprojlim_{0 < r < 1} W/k(r)W \cong \varprojlim_{0 < r < 1} W/k(r)V(1)$  and  $\varprojlim_{0 < r < 1} W/k(r)V(1) \cong \mathcal{W}$ . The latter isomorphism is given by the definition of the completion. Moreover, it extends to a unique homomorphism

$$k \otimes \iota: k \otimes_{k(1)} \varprojlim_{0 < r < 1} W/k(r)W \rightarrow k \otimes_{k(1)} \mathcal{V}(1) \cong \mathcal{V}$$

by the universality of a tensor product of  $k(1)$ -modules. It is an isomorphism of  $k$ -vector space because  $\iota$  is an isomorphism of  $k(1)$ -modules.  $\square$

## 1.2 Banach Algebras

A *seminorm* (resp. *norm*, *complete norm*) on a  $k$ -algebra  $\mathcal{A}$  is a seminorm (resp. norm, complete norm)  $\|\cdot\|_{\mathcal{A}}$  of the underlying  $k$ -vector space of  $\mathcal{A}$  satisfying the following conditions:

- (iv)  $\|fg\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}}\|g\|_{\mathcal{A}}$  for any  $(f, g) \in \mathcal{A} \times \mathcal{A}$ .
- (v)  $\|1\|_{\mathcal{A}} \in \{0, 1\}$ .

A seminorm  $\|\cdot\|_{\mathcal{A}}$  of  $\mathcal{A}$  is said to be *power-multiplicative* if it satisfies the following condition:

$$(iv)' \quad \|f^2\|_{\mathcal{A}} = \|f\|_{\mathcal{A}}^2 \text{ for any } f \in \mathcal{A}.$$

A seminorm  $\|\cdot\|_{\mathcal{A}}$  of  $\mathcal{A}$  is said to be *multiplicative* if it satisfies the following conditions:

$$(iv)'' \quad \|fg\|_{\mathcal{A}} = \|f\|_{\mathcal{A}}\|g\|_{\mathcal{A}} \text{ for any } (f, g) \in \mathcal{A} \times \mathcal{A}.$$

$$(v)' \quad \|1\|_{\mathcal{A}} = 1.$$

A *seminormed* (resp. *normed*, *Banach*)  $k$ -algebra is a pair  $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$  of a  $k$ -algebra  $\mathcal{A}$  and a seminorm (resp. norm, complete norm)  $\|\cdot\|_{\mathcal{A}}$  of  $\mathcal{A}$ . We call  $\mathcal{A}$  the underlying  $k$ -algebra of  $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ . We put  $\|\cdot\|_{(\mathcal{A}, \|\cdot\|_{\mathcal{A}})} := \|\cdot\|_{\mathcal{A}}$ , and call it the seminorm of  $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ .  $\mathcal{A}$  also denotes  $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ , and hence  $\|\cdot\|_{\mathcal{A}}$  is called the seminorm (resp. norm) of  $\mathcal{A}$ . Every normed (resp. Banach)  $k$ -algebra is an f-adic Tate ring (resp. a complete f-adic Tate ring) because  $k$  admits a topologically nilpotent unit and the closed unit disc is a ring of definition.

A Banach  $k$ -algebra  $\mathcal{A}$  is said to be *uniform* if its norm is power-multiplicative, and is said to be a *Banach function algebra* over  $k$  if the spectral radius function

$$\begin{aligned} \rho_{\mathcal{A}} : \mathcal{A} &\rightarrow [0, \infty) \\ a &\mapsto \inf_{n \in \mathbb{N}} \|a^n\|_{\mathcal{A}}^{\frac{1}{n}} \end{aligned}$$

is a complete norm. Obviously, every uniform Banach  $k$ -algebra is a Banach function algebra over  $k$ . We remark that  $\mathcal{A}$  is a Banach function algebra over  $k$  if and only if  $\|\cdot\|_{\mathcal{A}}$  is equivalent to  $\rho_{\mathcal{A}}$  as seminorms by Banach open mapping theorem ([Bou], I.3.3. Theorem 1). For an f-adic Tate ring  $A$ , we denote by  $A^{\circ} \subset A$  the subring of bounded elements. We emphasise that this convention is not compatible with that in [Ber1] 2.4. For the precise difference, see the following two lemmas.

**Lemma 1.7.** *For any Banach function algebra  $\mathcal{A}$  over  $k$ ,  $\mathcal{A}^{\circ}$  coincides with the subring consisting of elements  $a$  with  $\rho_{\mathcal{A}}(a) \leq 1$ .*

*Proof.* First, let  $a \in \mathcal{A}^{\circ}$ . Then  $\rho_{\mathcal{A}}(a) \leq 1$  by the definition of  $\rho_{\mathcal{A}}$ . Secondly, let  $a \in \mathcal{A}$  with  $\rho_{\mathcal{A}}(a) \leq 1$ . Then we have

$$\{\rho_{\mathcal{A}}(a^n) \mid n \in \mathbb{N}\} = \{\rho_{\mathcal{A}}(a)^n \mid n \in \mathbb{N}\} \subset [0, 1].$$

It implies  $a \in \mathcal{A}^{\circ}$  because  $\|\cdot\|_{\mathcal{A}}$  is equivalent to  $\rho_{\mathcal{A}}$ . □

**Lemma 1.8.** *Suppose that  $|k|$  is dense in  $[0, \infty)$ . For any Banach  $k$ -algebra  $\mathcal{A}$ ,  $(\mathcal{A}^{\circ})^{\text{ac}}$  coincides with the subring consisting of elements  $a$  with  $\rho_{\mathcal{A}}(a) \leq 1$ .*



*Proof.* Let  $a \in \mathcal{A}$ . Suppose  $\rho_{\mathcal{A}}(a) \leq 1$ . For any  $\epsilon \in k(1-)$ , we have  $\rho_{\mathcal{A}}(\epsilon a) = |\epsilon| \rho_{\mathcal{A}}(a) < 1$  and hence  $\epsilon a$  is topologically nilpotent. Therefore  $\epsilon a \in \mathcal{A}^\circ$ . It implies  $a \in (\mathcal{A}^\circ)^{\text{ac}}$ . On the other hand, suppose  $a \in (\mathcal{A}^\circ)^{\text{ac}}$ . Assume  $\rho_{\mathcal{A}}(a) > 1$ . Since  $|k|$  is dense in  $[0, \infty)$ , there is an  $\epsilon \in k(1-)$  such that  $\rho_{\mathcal{A}}(a)^{-1} < |\epsilon| < 1$ . However,  $\epsilon a \in \mathcal{A}^\circ$  and hence  $\rho_{\mathcal{A}}(\epsilon a) \leq 1$  by the definition of  $\rho_{\mathcal{A}}$ . It contradicts the inequality  $\rho_{\mathcal{A}}(\epsilon a) = |\epsilon| \rho_{\mathcal{A}}(a) > 1$ . Thus  $\rho_{\mathcal{A}}(a) \leq 1$ .  $\square$

**Proposition 1.9.** *Let  $\mathcal{A}$  be a Banach  $k$ -algebra. For any open integrally closed  $k(1)$ -subalgebra  $\mathcal{A}_0 \subset \mathcal{A}$ ,  $\mathcal{A}_0^{\text{ac}}$  is an open integrally closed  $k(1)$ -subalgebra.*

*Proof.* The openness is obvious because  $\mathcal{A}_0 \subset \mathcal{A}_0^{\text{ac}}$ . If the valuation of  $k$  is discrete, then  $\mathcal{A}_0^{\text{ac}} = \mathcal{A}_0$ . Therefore we may assume that  $|k|$  is dense in  $[0, \infty)$ . Let  $a \in \mathcal{A}$  be an element integral over  $\mathcal{A}_0^{\text{ac}}$ . Take a monic  $P \in \mathcal{A}_0^{\text{ac}}[T]$  with  $P(a) = 0$ . Let  $d \in \mathbb{N} \setminus \{0\}$  be the degree of  $P$ . For any  $\epsilon \in k(1-)$ , we have  $\epsilon^d P(T) \in \mathcal{A}_0[T]$  and hence  $\epsilon a$  is integral over  $\mathcal{A}_0$ . Since  $\mathcal{A}_0$  is integrally closed, we obtain  $\epsilon a \in \mathcal{A}_0$ . Thus  $a \in \mathcal{A}_0^{\text{ac}}$ . We conclude that  $\mathcal{A}_0^{\text{ac}}$  is integrally closed.  $\square$

**Lemma 1.10.** *Let  $\mathcal{A}$  be a  $k$ -algebra. For any adically separated lattice  $\mathcal{A}_0 \subset \mathcal{A}$  which is a subring,  $\|\cdot\|_{\mathcal{A}, \mathcal{A}_0}$  is a norm of  $\mathcal{A}$  as a  $k$ -algebra. Moreover, if  $\mathcal{A}_0$  is closed under square roots in  $\mathcal{A}$ , then  $\|\cdot\|_{\mathcal{A}, \mathcal{A}_0}$  is power-multiplicative.*

*Proof.* Since  $\mathcal{A}_0$  is an adically separated lattice,  $\|\cdot\|_0 := \|\cdot\|_{\mathcal{A}, \mathcal{A}_0}$  is a norm of  $\mathcal{A}$  as a  $k$ -vector space. Therefore it suffices to verify the submultiplicativity of  $\|\cdot\|_0$ . Let  $a, b \in \mathcal{A}$ . Assume  $\|ab\|_0 > \|a\|_0 \|b\|_0$ . Then none of  $a$  and  $b$  is zero. Suppose that the valuation of  $k$  is discrete. Then  $\mathcal{A}_0^{\text{ac}} = \mathcal{A}_0$  and the image of  $\|\cdot\|_0$  coincides with the image of the valuation of  $k$  by definition. Take  $c, d \in k^\times$  with  $|c| = \|a\|_0$  and  $|d| = \|b\|_0$ . Then  $|cd| = |c| |d| = \|a\|_0 \|b\|_0$  and hence  $|cd| < \|ab\|_0$ . Since  $\|c^{-1}a\|_0 = \|d^{-1}b\|_0 = 1$ , we have  $c^{-1}a, d^{-1}b \in \mathcal{A}_0$ , and it implies  $(cd)^{-1}ab = (c^{-1}a)(d^{-1}b) \in \mathcal{A}_0$  because  $\mathcal{A}_0$  is a subring. Therefore  $\|(cd)^{-1}ab\|_0 \leq 1$ . It contradicts the inequality  $|cd| < \|ab\|_0$ . Therefore  $|k|$  is dense in  $[0, \infty)$ . By the continuity of the multiplication  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , there are  $c, d \in k^\times$  such that  $\|a\|_0 < |c|$ ,  $\|b\|_0 < |d|$ , and  $|cd| < \|ab\|_0$ . We have  $\|c^{-1}a\|_0, \|d^{-1}b\|_0 < 1$  and hence  $c^{-1}a, d^{-1}b \in \mathcal{A}_0^{\text{ac}}$ . Since  $\mathcal{A}_0$  is a subring, so is  $\mathcal{A}_0^{\text{ac}}$ . Therefore we obtain  $(cd)^{-1}ab = (c^{-1}a)(d^{-1}b) \in \mathcal{A}_0^{\text{ac}}$ , and it implies  $\|(cd)^{-1}ab\|_0 \leq 1$ . It contradicts the inequality  $|cd| < \|ab\|_0$ . Thus  $\|ab\|_0 \leq \|a\|_0 \|b\|_0$ . We conclude that  $\|\cdot\|_0$  is a norm of  $\mathcal{A}$  as a  $k$ -algebra.

Suppose that  $\mathcal{A}_0$  is closed under square roots in  $\mathcal{A}$ . Then  $\mathcal{A}_0^{\text{ac}}$  is also closed under square roots in  $\mathcal{A}$  by the same argument in the proof of Proposition 1.9. Let  $a \in \mathcal{A}$ . If  $a = 0$ , then  $\|a^2\|_0 = \|a\|_0^2 = 0$ . Suppose  $a \neq 0$ . For any  $r \in (\|a^2\|_0, \infty)$ , there is a  $c \in k^\times$  such that  $\|a^2\|_0 \leq |c|^2 < r$  by the definition of  $\|\cdot\|_0$ . Then  $\|(c^{-1}a)^2\|_0 = |c|^{-2} \|a^2\|_0 \leq 1$ , and hence  $(c^{-1}a)^2 \in \mathcal{A}_0^{\text{ac}}$ . Since  $\mathcal{A}_0^{\text{ac}}$  is closed under square roots in  $\mathcal{A}$ , we obtain  $c^{-1}a \in \mathcal{A}_0^{\text{ac}}$  and hence  $\|c^{-1}a\|_0 \leq 1$ . It implies  $\|a\|_0^2 \leq |c|^2 < r$ . Thus  $\|a^2\|_0 = \|a\|_0^2$ . We conclude  $\|\cdot\|_0$  is power-multiplicative.  $\square$

**Proposition 1.11.** *A Banach  $k$ -algebra  $\mathcal{A}$  is a Banach function algebra over  $k$  (resp. a uniform Banach  $k$ -algebra) if and only if  $\mathcal{A}^\circ$  is bounded (resp. if  $\mathcal{A}(1) = \mathcal{A}^\circ$  and the closure of  $|k| \subset [0, \infty)$  contains  $\|\mathcal{A}^\circ\|$ ).*

*Proof.* The direct implication follows from Proposition 1.5 and Lemma 1.8. Suppose that  $\mathcal{A}(1) = \mathcal{A}^\circ$  and the closure of  $|k| \subset [0, \infty)$  contains  $\|\mathcal{A}^\circ\|$ . Since  $\mathcal{A}^\circ$  is closed under square roots,  $\|\cdot\|_{\mathcal{A}}$  is power-multiplicative by Proposition 1.5 and Lemma 1.10.

Suppose that  $\mathcal{A}^\circ$  is bounded. Take a  $c \in k^\times$  such that  $\mathcal{A}^\circ \subset c\mathcal{A}(1)$ . Since  $(c\mathcal{A}(1))^{\text{ac}} = c\mathcal{A}(1)^{\text{ac}} = c\mathcal{A}(1)$ , we have  $(\mathcal{A}^\circ)^{\text{ac}} \subset (c\mathcal{A}(1))^{\text{ac}} = c\mathcal{A}(1)$ . Let  $a \in (\mathcal{A}^\circ)^{\text{ac}}$ . We have  $a^n \in (\mathcal{A}^\circ)^{\text{ac}}$  for any  $n \in \mathbb{N}$  by Lemma 1.10. It implies  $\{a^n \mid n \in \mathbb{N}\} \subset c\mathcal{A}(1)$ , and hence  $a \in \mathcal{A}^\circ$ . We obtain  $(\mathcal{A}^\circ)^{\text{ac}} = \mathcal{A}^\circ$ . By Lemma 1.7,  $\mathcal{A}^\circ$  coincides with the closed unit ball with respect to  $\rho_{\mathcal{A}}$ . Since  $\mathcal{A}^\circ$  is an open bounded subspace of  $\mathcal{A}$ ,  $\rho_{\mathcal{A}}$  is equivalent to  $\|\cdot\|_{\mathcal{A}}$ . We conclude that  $\mathcal{A}$  is a Banach function algebra over  $k$ .  $\square$

**Corollary 1.12.** *For any Banach function algebra  $\mathcal{A}$  over  $k$ ,  $\mathcal{A}^\circ$  is adically closed and coincides with the closed unit ball with respect to the spectral radius function.*

*Proof.* It is a direct consequence of Lemma 1.7 and Proposition 1.11.  $\square$

**Corollary 1.13.** *Suppose that  $|k|$  is dense in  $[0, \infty)$ . For any Banach function algebra  $\mathcal{A}$  over  $k$ ,  $\|\cdot\|_{\mathcal{A}, \mathcal{A}^\circ}$  is a unique power-multiplicative norm of  $\mathcal{A}$  equivalent to  $\|\cdot\|_{\mathcal{A}}$ .*

*Proof.* By Proposition 1.11 and Corollary 1.12,  $\mathcal{A}^\circ$  is bounded and adically closed, and hence  $\|\cdot\|_{\mathcal{A}, \mathcal{A}^\circ}$  is equivalent to  $\|\cdot\|_{\mathcal{A}}$ . It is a norm of  $\mathcal{A}$  as a  $k$ -algebra and is power-multiplicative by Lemma 1.10, because  $(\mathcal{A}^\circ)^{\text{ac}} = \mathcal{A}^\circ$  is closed under square roots. The uniqueness is obvious because every power-multiplicative norm equivalent to  $\|\cdot\|_{\mathcal{A}}$  coincides with the spectral radius function  $\rho_{\mathcal{A}}$  by [Ber1] 1.3.1. Theorem.  $\square$

## 2 Uniformity of Berkovich Spectra

We note that the valuation of the base field of  $k$  is assumed to be non-trivial throughout this paper. We recall the notion of a rational localisation of the Berkovich spectrum of a Banach  $k$ -algebra in §2.1. We construct a uniform Banach  $k$ -algebra whose rational localisation is not uniform in §2.2. We also give several affirmative results for a rational localisation of a Berkovich spectrum in §2.3.

### 2.1 Rational Localisations

Let  $\mathcal{A}$  be a Banach  $k$ -algebra. A seminorm  $x$  on the underlying  $k$ -algebra  $\mathcal{A}$  is said to be bounded if  $x(f) \leq \|f\|_{\mathcal{A}}$  for any  $f \in \mathcal{A}$ . We denote by  $\mathcal{M}(\mathcal{A})$  the set of bounded multiplicative seminorms endowed with the strongest topology with respect to which for each  $f \in \mathcal{A}$ , the associated map

$$\begin{aligned} |f|^* : \mathcal{M}(\mathcal{A}) &\rightarrow [0, \infty) \\ x &\rightarrow x(f) \end{aligned}$$

is continuous.  $\mathcal{M}(\mathcal{A})$  is a compact Hausdorff space, and  $\mathcal{M}(A) \neq \emptyset$  if and only if  $\mathcal{A} \neq 0$ . We call  $\mathcal{M}(\mathcal{A})$  the *Berkovich spectrum* of  $\mathcal{A}$ . For each  $(f, x) \in \mathcal{A} \times \mathcal{M}(\mathcal{A})$ , we put  $|f(x)| := x(f)$ .

Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach  $k$ -algebras. For a bounded homomorphism  $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ , we denote by  $\mathcal{M}(\varphi): \mathcal{M}(\mathcal{B}) \rightarrow \mathcal{M}(\mathcal{A})$  the associated continuous map sending an  $x \in \mathcal{M}(\mathcal{B})$  to  $\varphi(x) := x \circ \varphi \in \mathcal{M}(\mathcal{A})$ . The correspondences  $\mathcal{A} \rightsquigarrow \mathcal{M}(\mathcal{A})$  and  $\varphi \rightsquigarrow \mathcal{M}(\varphi)$  give a contravariant functor  $\mathcal{M}$  from the category of Banach  $k$ -algebras and bounded  $k$ -algebra homomorphisms to the category of topological spaces and continuous maps.

Let  $\mathcal{A}$  be a Banach  $k$ -algebra. For an  $n \in \mathbb{N} \setminus \{0\}$  and  $r_1, \dots, r_n \in (0, \infty)$ , we denote by  $\mathcal{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$  the  $k$ -subalgebra of  $\mathcal{A}[[T_1, \dots, T_n]]$  consisting of elements  $f = \sum_{(h_i)_{i=1}^n \in \mathbb{N}^n} f_{(h_1, \dots, h_n)} T_1^{h_1} \cdots T_n^{h_n}$  with  $\lim_{h_1 + \dots + h_n \rightarrow \infty} \|f_{(h_1, \dots, h_n)}\| r_1^{h_1} \cdots r_n^{h_n} = 0$ . We endow it with the complete norm

$$\begin{aligned} \|\cdot\|_{\mathcal{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}} : \mathcal{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} &\rightarrow [0, \infty) \\ f = \sum_{(h_i)_{i=1}^n \in \mathbb{N}^n} f_{(h_1, \dots, h_n)} T_1^{h_1} \cdots T_n^{h_n} &\rightarrow \sup_{(h_i)_{i=1}^n \in \mathbb{N}^n} \|f_{(h_1, \dots, h_n)}\| r_1^{h_1} \cdots r_n^{h_n}. \end{aligned}$$

We call  $\|\cdot\|_{\mathcal{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}}$  the Gauss norm of radius  $(r_1, \dots, r_n)$ . In particular,  $(r_1, \dots, r_n) = (1, \dots, 1)$ , we put  $\mathcal{A}\{T_1, \dots, T_n\} := \mathcal{A}\{1^{-1}T_1, \dots, 1^{-1}T_n\}$ .

For  $f_0, f_1, \dots, f_n \in \mathcal{A}$  with  $\sum_{i=1}^n \mathcal{A}f_i = \mathcal{A}$ , we denote by  $\mathcal{A}\{r_1^{-1}f_0^{-1}f_1, \dots, r_n^{-1}f_0^{-1}f_n\}$  the quotient of  $\mathcal{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$  by the closure of the ideal generated by  $\{f_0T_i - f_i \mid i \in \mathbb{N} \cap [1, n]\}$ . In particular, we put

$$\begin{aligned} \mathcal{A}\{f_0^{-1}f_1, \dots, f_0^{-1}f_n\} &:= \mathcal{A}\{1^{-1}f_0^{-1}f_1, \dots, 1^{-1}f_0^{-1}f_n\} \\ \mathcal{A}\{r_1^{-1}f_1, \dots, r_n^{-1}f_n\} &:= \mathcal{A}\{r_1^{-1}1^{-1}f_1, \dots, r_n^{-1}1^{-1}f_n\} \\ \mathcal{A}\{f_1, \dots, f_n\} &:= \mathcal{A}\{1^{-1}1^{-1}f_1, \dots, 1^{-1}1^{-1}f_n\} \\ \mathcal{A}\{f_1, \dots, f_{n-1}\} &:= \mathcal{A}\{f_1, \dots, f_{n-1}, 1\} \end{aligned}$$

We set  $\mathcal{M}(\mathcal{A})\{r_1^{-1}f_0^{-1}f_1, \dots, r_n^{-1}f_0^{-1}f_n\} \subset \mathcal{M}(\mathcal{A})$  the closed subset of points  $x$  with  $|f_i(x)| \leq r_i|f_0(x)|$  for any  $i \in \mathbb{N} \cap [1, n]$ . A subset of  $\mathcal{M}(\mathcal{A})$  of such a form is called a rational domain. The continuous map  $\mathcal{M}(\mathcal{A}\{r_1^{-1}f_0^{-1}f_1, \dots, r_n^{-1}f_0^{-1}f_n\}) \rightarrow \mathcal{M}(\mathcal{A})$  associated to the canonical submetric homomorphism  $\mathcal{A} \rightarrow \mathcal{A}\{r_1^{-1}f_0^{-1}f_1, \dots, r_n^{-1}f_0^{-1}f_n\}$  is a homeomorphism onto  $\mathcal{M}(\mathcal{A})\{r_1^{-1}f_0^{-1}f_1, \dots, r_n^{-1}f_0^{-1}f_n\}$ .

The underlying  $k$ -algebra of the Banach  $k$ -algebra  $k\{r_1^{-1}T_1, \dots, r_m^{-1}T_m\}$  is Noetherian, and hence every ideal of it is closed for any  $n \in \mathbb{N} \setminus \{0\}$  and  $r_1, \dots, r_m \in (0, \infty)^m$ . If  $\mathcal{A}$  is isomorphic to the quotient of such a Banach  $k$ -algebra, the isomorphism class of  $\mathcal{A}\{f_0^{-1}f_1, \dots, f_0^{-1}f_n\}$  for an  $n \in \mathbb{N} \setminus \{0\}$  and  $f_0, f_1, \dots, f_n \in \mathcal{A}$  with  $\sum_{i=1}^n \mathcal{A}f_i = \mathcal{A}$  depends only on the rational domain  $\mathcal{M}(\mathcal{A})\{f_0^{-1}f_1, \dots, f_0^{-1}f_n\}$ , and hence is denoted by  $\mathcal{O}_{\mathcal{M}(\mathcal{A})}(\mathcal{M}(\mathcal{A})\{f_0^{-1}f_1, \dots, f_0^{-1}f_n\})$ . The correspondence  $U \rightsquigarrow \mathcal{O}_{\mathcal{M}(\mathcal{A})}(U)$  gives a presheaf

of complete topological  $k$ -algebras on the Grothendieck topology generated by rational subsets of  $\mathcal{M}(\mathcal{A})$ . We remark that there is no canonical norm of  $\mathcal{O}_{\mathcal{M}(\mathcal{A})}(U)$  when  $\mathcal{A}$  is not reduced, and hence we have to forget the norm on each section.

On the other hand, if  $\mathcal{A}$  is not isomorphic to a quotient of the Banach  $k$ -algebra of the form  $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ , the isomorphism class of  $\mathcal{A}\{f_0^{-1}f_1, \dots, f_0^{-1}f_n\}$  for an  $n \in \mathbb{N} \setminus \{0\}$  and  $f_0, f_1, \dots, f_n \in \mathcal{A}$  with  $\sum_{i=1}^n \mathcal{A}f_i = \mathcal{A}$  depends on the presentation  $(n, (f_0, f_1, \dots, f_n))$  of  $\mathcal{M}(\mathcal{A})\{f_0^{-1}f_1, \dots, f_0^{-1}f_n\}$ , and hence the presheaf  $\mathcal{O}_{\mathcal{M}(\mathcal{A})}$  is not well-defined.

We recall the relation between rational localisations of a Berkovich spectrum and an adic spectrum. For simplicity, we only deal with a Weierstrass localisation. For an  $a \in \mathcal{A}$ , we define

$$\begin{aligned}\mathcal{A}(1)\langle a \rangle &:= \varprojlim_{0 < r < 1} \mathcal{A}(1)[a]/\mathcal{A}(r)[a] \\ \mathcal{A}\langle a \rangle &:= k \otimes_{k(1)} \mathcal{A}(1)\langle a \rangle,\end{aligned}$$

where  $\mathcal{A}(r)[a] \subset \mathcal{A}$  denotes the smallest subset closed under the addition, the multiplication, and the scalar multiplication containing  $\mathcal{A}(r) \cup \{a\}$  for each  $r \in (0, 1)$ . By Lemma 1.6,  $\mathcal{A}(1)\langle a \rangle$  is naturally isomorphic to  $\varprojlim_{0 < r < 1} \mathcal{A}(1)[a]/k(r)\mathcal{A}(1)[a]$  as topological  $k(1)$ -modules. Therefore its underlying  $k(1)$ -module is torsion-free and hence is flat. It implies that the canonical homomorphism  $\mathcal{A}(1)\langle a \rangle \rightarrow \mathcal{A}\langle a \rangle$  is injective, and we identify  $\mathcal{A}(1)\langle a \rangle$  with its image. We equip  $\mathcal{A}\langle a \rangle$  with  $\|\cdot\|_{\mathcal{A}\langle a \rangle} := \|\cdot\|_{\mathcal{A}\langle a \rangle, \mathcal{A}(1)\langle a \rangle}$ . By Lemma 1.10,  $\|\cdot\|_{\mathcal{A}\langle a \rangle}$  is a norm of  $\mathcal{A}\langle a \rangle$  as a  $k$ -algebra, and  $\mathcal{A}\langle a \rangle$  is a Banach  $k$ -algebra. The norm topology restricted on  $\mathcal{A}\langle a \rangle(1)$  coincides with the inverse limit topology. The continuous map  $\mathcal{M}(\mathcal{A}\langle a \rangle) \rightarrow \mathcal{M}(\mathcal{A})$  associated to the canonical submetric homomorphism  $\mathcal{A} \rightarrow \mathcal{A}\langle a \rangle$  is a homeomorphism onto the closed subset  $\mathcal{M}(\mathcal{A})\langle a \rangle := \{x \in \mathcal{M}(\mathcal{A}) \mid |a(x)| \leq 1\}$ , and the image of  $\mathcal{A}$  is dense in  $\mathcal{A}\langle a \rangle$ .

**Proposition 2.1.** *Let  $\mathcal{A}$  be a Banach  $k$ -algebra, and  $a \in \mathcal{A}$ . Then the natural bounded homomorphism  $\mathcal{A}\{T\} \rightarrow \mathcal{A}\langle a \rangle$  sending  $T$  to the image of  $a$  is an admissible surjective homomorphism, and its kernel is the closure  $(T - a)^\wedge$  of the principal ideal  $(T - a) \subset \mathcal{A}\{T\}$  generated by  $T - a$ . Moreover, it induces an isometric isomorphism  $\mathcal{A}\{a\} \rightarrow \mathcal{A}\langle a \rangle$ .*

*Proof.* Let  $\iota: \mathcal{A} \rightarrow \mathcal{A}\langle a \rangle$  denote the canonical bounded homomorphism. It extends to a unique bounded homomorphism  $\iota\{T = a\}: \mathcal{A}\{T\} \rightarrow \mathcal{A}\langle a \rangle$  sending  $T$  to  $\iota(a)$  because  $\|\iota(a)\|_{\mathcal{A}\langle a \rangle} \leq 1$ . For any  $f \in \mathcal{A}(1)\langle a \rangle$ , there is a sequence  $(f_i)_{i \in \mathbb{N}} \in \prod_{i=0}^{\infty} \mathcal{A}(2^{-i})[a]$  such that  $f = \sum_{i=0}^{\infty} \iota(f_i)$  by the definition of the topology of  $\mathcal{A}\langle a \rangle$ . Take a lift  $F_i \in \mathcal{A}[T]$  of  $f_i$  each of whose coefficient lies in  $\mathcal{A}(2^{-i})$  for each  $i \in \mathbb{N}$ . By the completeness of  $\mathcal{A}$ ,  $F := \sum_{i=0}^{\infty} F_i$  converges in  $\mathcal{A}\{T\}$ . By the continuity of  $\iota\{T = a\}$ ,  $\iota\{T = a\}(F) = \sum_{i=0}^{\infty} \iota\{T = a\}(F_i) = \sum_{i=0}^{\infty} \iota(f_i) = f$ . Therefore  $\iota\{T = a\}(\mathcal{A}\{T\}(1)) = \mathcal{A}\langle a \rangle(1)$ , and  $\iota\{T = a\}$  is surjective. By Banach open mapping theorem ([Bou], I.3.3. Theorem 1), the image of  $\mathcal{A}\{T\}(1)$  by  $\iota\{T = a\}$  coincides with  $\mathcal{A}(1)\langle a \rangle$  because  $\iota\{T = a\}(\mathcal{A}(1)[T]) = \iota(\mathcal{A}(1)[a])$  is dense in the clopen subset  $\mathcal{A}(1)\langle a \rangle$ . Moreover,  $\iota\{T = a\}$  is submetric because the

image of the dense  $k(1)$ -subalgebra  $\mathcal{A}(1)[T]$  of  $\mathcal{A}\{T\}(1)$  by  $\iota\{T = a\}$  coincides with  $\iota(\mathcal{A}(1)[a]) \subset \mathcal{A}(1)\langle a \rangle$ . The equality  $\iota\{T = a\}(F) \in k(r)\mathcal{A}(1)\langle a \rangle$  by the definition of  $\|\cdot\|_{\mathcal{A}\langle a \rangle}$ . Take an  $N \in \mathbb{N}$  such that  $\|F_i\|_{\mathcal{A}} < r$  for any  $i > N$ . Set  $G := \sum_{i=0}^N F_i T^i \in \mathcal{A}[T]$ . Since  $\iota\{a\}$  is submetric, we have  $\iota\{a\}(F - G) \in \mathcal{A}\langle a \rangle(r-) \subset k(r)\mathcal{A}(1)\langle a \rangle$  and hence  $\iota(G(a)) = \iota\{a\}(G) = \iota\{a\}(F) - \iota\{a\}(F - G) \in k(r)\mathcal{A}(1)\langle a \rangle$ . Therefore we obtain  $G(a) \in \mathcal{A}(r)[a]$  by the definition of  $\iota$ , and there is an  $H \in \mathcal{A}[T]$  such that  $H(a) = G(a)$  and each coefficient of  $H$  lies in  $\mathcal{A}(r)$ . We get

$$F = (F - G) + (G - G(a)) - (H - H(a)) + H \in \mathcal{A}\{T\}(r) + (T - a),$$

and hence  $\|F + (T - a)\|_{\mathcal{A}\langle a \rangle} \leq r$ . It implies  $\|F + (T - a)\|_{\mathcal{A}\langle a \rangle} = \|\varphi(F)\|_{\mathcal{A}\langle a \rangle}$ . Thus  $\iota\{a\}$  is isometric. Since  $\iota\{a\}$  is surjective, and it is an isometric isomorphism.  $\square$

**Corollary 2.2.** *Let  $\mathcal{A}$  be a Banach  $k$ -algebra, and  $a \in \mathcal{A}$ . Then  $\mathcal{A}(1)\langle a \rangle = \mathcal{A}\langle a \rangle(1)$ , and hence  $\mathcal{A}(1)\langle a \rangle$  is adically closed.*

**Proposition 2.3.** *Let  $\mathcal{A}$  be a uniform Banach  $k$ -algebra, and  $a \in \mathcal{A}$ . Then  $(T - a) \subset \mathcal{A}\{T\}$  is a closed ideal, and the canonical bounded homomorphism  $\mathcal{A}\{T\}/(T - a) \rightarrow \mathcal{A}\langle a \rangle$  is an isometric isomorphism.*

*Proof.* We have only to verify the closedness of  $(T - a)$ . Let  $F \in \mathcal{A}\{T\}$ . For each  $x \in \mathcal{M}(\mathcal{A})$ , denote by  $\tilde{x} \in \mathcal{M}(\mathcal{A}\{T\})$  the point given by setting  $|G(\tilde{x})| = \sup_{i=0}^{\infty} |G_i(\tilde{x})|$  for each  $G = \sum_{i=0}^{\infty} G_i T^i \in \mathcal{A}\{T\}$ . Then  $|(T - a)(\tilde{x})| = \max\{1, |a(x)|\} \geq 1$ , and hence  $|F(\tilde{x})| \leq |(T - a)(\tilde{x})| |F(\tilde{x})| = |(T - a)F(\tilde{x})| \leq \|T - a\|_{\mathcal{A}\{T\}} \|F\|_{\mathcal{A}\{T\}}$  for any  $x \in \mathcal{M}(\mathcal{A})$ . The uniformity of  $\mathcal{A}$  implies  $\|F\|_{\mathcal{A}\{T\}} \leq \|(T - a)F\|_{\mathcal{A}\{T\}} \leq \|T - a\|_{\mathcal{A}\{T\}} \|F\|_{\mathcal{A}\{T\}}$ . Therefore the multiplication  $(T - a) \times: \mathcal{A}\{T\} \rightarrow \mathcal{A}\{T\}$  is an admissible injective homomorphism, and its image  $(T - a)$  is closed.  $\square$

## 2.2 Negative Facts

We give an example of a rational localisation which does not preserve the uniformity. We set  $a_0 := 1$ . For each  $i \in \mathbb{N} \setminus \{0\}$ , we denote by  $a_i \in \mathbb{N}$  the smallest integer greater than  $\log_2 i$ . For an  $r \in (0, \infty)$ , we denote by  $k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\} \subset k\{r^{-1}X, U\}$  the closure of the  $k$ -subalgebra generated by  $\{U^i X^{a_i} \mid i \in \mathbb{N}\}$ . Since  $k\{r^{-1}X, U\}$  is uniform, so is  $k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}$ .

**Theorem 2.4.** *For any  $r \in (1, \infty)$ , the rational localisation  $k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}\{X\}$  is not a Banach function algebra over  $k$ .*

*Proof.* To begin with, we show that the natural inclusion

$$\iota: k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\} \hookrightarrow k\{r^{-1}X, U\} \hookrightarrow k\{U\}[[X]]$$

is uniquely extended to a continuous homomorphism

$$\iota\{X\} : k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{X\} \hookrightarrow k\{U\}[[X]]$$

with respect to the inverse limit topology of  $k\{U\}[[X]]$ . The inclusion  $\iota$  is continuous because it is the restriction of the natural inclusion  $k\{r^{-1}X, U\} \hookrightarrow k\{U\}[[X]]$  given through the isometric isomorphism  $k\{r^{-1}X, U\} \cong k\{U\}\{r^{-1}X\}$ . The  $k$ -algebra homomorphism  $k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}[T] \rightarrow k\{U\}[[X]]$  extending  $\iota$  and sending  $T$  to  $X$  uniquely extends to a continuous homomorphism

$$\iota\{T = X\} : k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{T\} \rightarrow k\{U\}[[X]]$$

because  $X$  is a topologically nilpotent element and  $k\{U\}[[X]]$  is a Fréchet  $k$ -algebra. Its kernel contains the principal ideal  $(T - X)$ , and hence it induces a continuous homomorphism

$$\iota\{X\} : k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{X\} \rightarrow k\{U\}[[X]]$$

by Proposition 2.3. Put  $b_0 := a_0 - 1 = 0$  and  $b_i = a_i$  for each  $i \in \mathbb{N}$ . Let  $f = \sum_{h=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=b_i}^{\infty} f_{h,i,j} U^i X^j T^h \in \ker(\iota\{T = X\})$ . Since  $\lim_{h+i+j \rightarrow \infty} |f_{h,i,j}| r^j = 0$ , the infinite sum  $\sum_{i=0}^{\infty} \sum_{l=a_i}^{\infty} \sum_{j=b_i}^l f_{l-j,i,j} U^i X^j T^{l-j}$  also converges to  $f$ . Therefore by the continuity of  $\iota\{T = X\}$ , we have

$$0 = \iota\{T = X\}(f) = \sum_{i=0}^{\infty} \sum_{l=a_i}^{\infty} \left( \sum_{j=b_i}^l f_{l-j,i,j} \right) U^i X^l.$$

Comparing the coefficients of both sides, we obtain  $0 = \sum_{j=b_i}^l f_{l-j,i,j}$  for any  $i, l \in \mathbb{N}$  with  $l \geq b_i$ . In particular,  $f_{0,i,b_i} = 0$  for any  $i \in \mathbb{N}$ . Set

$$g = \sum_{h=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=b_i}^{\infty} \left( - \sum_{m=0}^h f_{h-m,i,j+m+1} \right) U^i X^j T^h \in k[[U, X, T]].$$

The equality  $\lim_{h+i+j \rightarrow \infty} |f_{h,i,j}| r^j = 0$  implies

$$\lim_{h \rightarrow \infty} \left( \sup_{i \in \mathbb{N}} \sup_{j \geq b_i} \left| - \sum_{m=0}^h f_{h-m,i,j+m+1} \right| r^j \right) = 0$$

because  $r > 1$ , and hence  $g$  lies in  $k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{T\}$ . We have

$$\begin{aligned} (T - X)g &= \sum_{i=0}^{\infty} \sum_{j=b_i}^{\infty} f_{0,i,j+1} U^i X^{j+1} - \sum_{h=1}^{\infty} \sum_{i=0}^{\infty} \sum_{m=0}^{h-1} f_{h-1-m,i,b_i+m+1} T (U^i X^{b_i} T^{h-1}) \\ &\quad + \sum_{h=1}^{\infty} \sum_{i=0}^{\infty} \sum_{j=b_i+1}^{\infty} \left( - \sum_{m=0}^{h-1} f_{h-1-m,i,j+m+1} + \sum_{m=0}^h f_{h-m,i,j+m} \right) U^i X^j T^h \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty} \sum_{j=b_i+1}^{\infty} f_{0,i,j} U^i X^j - \sum_{h=1}^{\infty} \sum_{i=0}^{\infty} \sum_{m=0}^{h-1} f_{h-1-m,i,b_i+m+1} U^i X^{b_i} T^h \\
&\quad + \sum_{h=1}^{\infty} \sum_{i=0}^{\infty} \sum_{j=b_i+1}^{\infty} f_{h,i,j} U^i X^j T^h \\
&= \sum_{i=0}^{\infty} \left( f_{0,i,b_i} U^i X^{b_i} + \sum_{j=b_i+1}^{\infty} f_{0,i,j} U^i X^j \right) \\
&\quad + \sum_{h=1}^{\infty} \sum_{i=0}^{\infty} \left( \sum_{j=b_i}^{b_i+h} f_{b_i+h-j,i,j} - \sum_{m=0}^{h-1} f_{h-1-m,i,b_i+m+1} \right) U^i X^{b_i} T^h \\
&\quad + \sum_{h=1}^{\infty} \sum_{i=0}^{\infty} \sum_{j=b_i+1}^{\infty} f_{h,i,j} U^i X^j T^h \\
&= \sum_{i=0}^{\infty} \sum_{j=b_i}^{\infty} f_{0,i,j} U^i X^j + \sum_{h=1}^{\infty} \sum_{i=0}^{\infty} f_{h,i,b_i} U^i X^{b_i} T^h \\
&\quad + \sum_{h=1}^{\infty} \sum_{i=0}^{\infty} \sum_{j=b_i+1}^{\infty} f_{h,i,j} U^i X^j T^h \\
&= \sum_{h=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=b_i}^{\infty} f_{h,i,j} U^i X^j T^h = f.
\end{aligned}$$

Therefore  $f \in (T - X)$ . We obtain  $\ker(\iota(T - X)) = (T - X)$ , and hence  $\iota\{X\}$  is injective.

Since the image of  $k[U^i X^{a_i} \mid i \in \mathbb{N}]$  is dense in  $k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}\{X\}$ , the image of  $\iota\{X\}$  is contained in the closed  $k$ -subalgebra

$$\mathcal{A} := \left\{ F = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} F_{i,j} U^i X^j \in k\{U\}[[X]] \mid F_{i,j} = 0, \forall j < b_i \right\}.$$

Denote by  $\varphi: k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\} \rightarrow k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}\{X\}$  the canonical bounded homomorphism. We verify  $\|\varphi(U^i X^j)\|_{k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}\{X\}} = r^{b_i}$  for any  $i, j \in \mathbb{N}$  with  $j \geq b_i$ . Let  $i_0, j_0 \in \mathbb{N}$  with  $j_0 \geq b_{i_0}$ . We have

$$\begin{aligned}
&\|\varphi(U^{i_0} X^{j_0})\|_{k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}\{X\}} \\
&\leq \|\varphi(X^{j_0-b_{i_0}})\|_{k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}\{X\}} \|\varphi(U^{i_0} X^{b_{i_0}})\|_{k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}\{X\}} \leq r^{b_{i_0}}.
\end{aligned}$$

Assume  $\|\varphi(U^{i_0} X^{j_0})\|_{k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}\{X\}} < r^{b_{i_0}}$ . Then there are an  $F \in k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}\{T\}$  and a  $G \in k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}\{T\}(r^{b_{i_0}} -)$  such that  $U^{i_0} X^{j_0} = (T - X)F + G$ . Set

$$F = \sum_{h=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=b_i}^{\infty} F_{h,i,j} U^i X^j T^h$$

$$G = \sum_{h=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=b_i}^{\infty} G_{h,i,j} U^i X^j T^h.$$

We have

$$\begin{aligned} U^{i_0} X^{j_0} &= -X \sum_{i=0}^{\infty} \sum_{j=b_i}^{\infty} F_{0,i,j} U^i X^j + \sum_{i=0}^{\infty} \sum_{j=b_i}^{\infty} G_{0,i,j} U^i X^j \\ &= \sum_{i=0}^{\infty} \left( G_{0,i,b_i} U^i X^{b_i} + \sum_{j=b_i+1}^{\infty} (-F_{0,i,j-1} + G_{0,i,j}) U^i X^j \right) \end{aligned}$$

and

$$\begin{aligned} 0 &= \sum_{i=0}^{\infty} F_{h-1,i,j} U^i X^{b_i} + \sum_{i=0}^{\infty} \sum_{j=b_i+1}^{\infty} (F_{h-1,i,j} - F_{h,i,j-1}) U^i X^j \\ &\quad + \sum_{i=0}^{\infty} \sum_{j=b_i}^{\infty} G_{h,i,j} U^i X^j \\ &= \sum_{i=0}^{\infty} (F_{h-1,i,b_i} + G_{h,i,b_i}) U^i X^{b_i} + \sum_{i=0}^{\infty} \sum_{j=b_i+1}^{\infty} (F_{h-1,i,j} - F_{h,i,j-1} + G_{h,i,j}) U^i X^j \end{aligned}$$

for any  $h \in \mathbb{N} \setminus \{0\}$ .

Assume  $j_0 = b_{i_0}$ . Comparing the coefficients of both sides through the inclusion  $\iota\{z\}$ , we obtain  $G_{0,i_0,b_{i_0}} = 1$ . It contradicts the inequality  $\|G_{0,i_0,b_{i_0}} U^{i_0} X^{b_{i_0}}\|_{k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}} \leq \|G\|_{k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}\{T\}} < r^{b_{i_0}}$ . Therefore we get  $j_0 > b_{i_0}$ .

Comparing the coefficients of both sides of the previous equalities again, we obtain  $G_{0,i,b_i} = 0$  for any  $i \in \mathbb{N}$ ,  $F_{0,i_0,j_0-1} + 1 = G_{0,i_0,j_0}$ ,  $F_{0,i,j-1} = G_{0,i,j}$  for any  $i, j \in \mathbb{N}$  with  $j \geq b_i + 1$ ,  $F_{h-1,i,b_i} + G_{h,i,b_i} = 0$  for any  $h, i, j \in \mathbb{N}$  with  $h \geq 1$ , and  $F_{h-1,i,j} - F_{h,i,j-1} + G_{h,i,j} = 0$  for any  $h, i, j \in \mathbb{N}$  with  $h \geq 1$  and  $j \geq b_i + 1$ . We have

$$\begin{aligned} |G_{h,i_0,j_0}| &= \|G_{h,i_0,j_0} U^{i_0} X^{j_0} T^h\|_{k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}} r^{-j_0} \\ &\leq \|G\|_{k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}\{T\}} r^{-j_0} < r^{-(j_0-b_0)} < 1 \end{aligned}$$

for any  $h \in \mathbb{N}$ , and hence

$$\begin{aligned} |F_{0,i_0,j_0-1}| &= |G_{0,i_0,j_0} - 1| = 1 \\ |F_{1,i_0,j_0-1}| &= |F_{0,i_0,j_0} + G_{1,i_0,j_0}| = 1 \\ &\vdots \\ |F_{h,i_0,j_0-1}| &= |F_{h-1,i_0,j_0} + G_{h,i_0,j_0}| = 1 \end{aligned}$$

for any  $h \in \mathbb{N} \setminus \{0\}$ . It contradicts the fact  $\lim_{h \rightarrow \infty} (\sup_{i=0}^{\infty} \sup_{j=b_i}^{\infty} |F_{h,i,j}| r^j) = 0$ . Thus  $\|\varphi(U^{i_0} X^{j_0})\|_{k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\}\{X\}} = r^{b_{i_0}}$ .



We obtain

$$\lim_{n \rightarrow \infty} \|\varphi(UX)^n\|_{k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{X\}}^{\frac{1}{n}} = \lim_{n \rightarrow \infty} r^{\frac{b_n}{n}} = r^{\lim_{n \rightarrow \infty} \frac{b_n}{n}} = 1.$$

Since the valuation of  $k$  is non-trivial, there is a  $c \in k^\times$  such that  $|c| < 1$ . Take an  $N \in \mathbb{N} \setminus \{0\}$  with  $|c|r^N > 1$ . Denote by  $d_i \in \mathbb{N}$  the greatest integer satisfying  $Nd_i \geq b_i$  for each  $i \in \mathbb{N}$ . The infinite sum  $\sum_{i=0}^{\infty} c^{d_i}(UX)^{Ni}$  is a Cauchy sequence with respect to the seminorm of  $k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{X\}$  given by the spectral radius function. On the other hand, it does not converges because

$$\lim_{i \rightarrow \infty} \|c^{d_i}(UX)^i\|_{k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{X\}} = \lim_{i \rightarrow \infty} |c|^{d_i} r^{b_i} \geq \lim_{i \rightarrow \infty} \left(|c|^{\frac{1}{N}} r\right)^{b_i} = \infty.$$

Thus  $k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{X\}$  is not a Banach function algebra over  $k$ .  $\square$

**Corollary 2.5.** *For any  $r \in (1, \infty)$ , the Banach  $k$ -algebras  $k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{X\}$  and  $k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{X, UX\}$  do not share the isomorphism classes, while they correspond to the same rational domain of  $\mathcal{M}(k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\})$ . In particular, the uniform Banach  $k$ -algebra  $k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}$  is not sheafy.*

### 2.3 Affirmative Facts

We give two affirmative facts about rational localisations and the sheaf condition for Berkovich spectra.

**Definition 2.6.** *A Banach  $k$ -algebra  $\mathcal{A}$  is said to be spectrally reduced if the spectral radius of every non-zero element of  $\mathcal{A}$  is non-zero.*

A Banach  $k$ -algebra is spectrally reduced if and only if the Gel'fand transform of  $\mathcal{A}$  on the Berkovich spectrum is injective. The underlying  $k$ -algebra of a spectrally reduced Banach  $k$ -algebra is always reduced. Obviously, a Banach function algebra over  $k$  is spectrally reduced. Conversely, a reduced affinoid  $k$ -algebra is a Banach function algebra over  $k$ .

**Definition 2.7.** *A Banach  $k$ -algebra is said to be a Banach field over  $k$  if its underlying  $k$ -algebra is a field.*

We remark that a Banach field over  $k$  is not necessarily an extension of  $k$  as complete valuation fields. The norm of a Banach field over  $k$  is just submultiplicative, but is not multiplicative. Such Banach  $k$ -algebras are significant because they appear when we consider the Banach  $k$ -algebra of sections on a closed subset of the Berkovich spectrum associated to a Banach  $k$ -algebra topologically of finite type.

**Theorem 2.8.** *Let  $\mathcal{A}$  be a uniform Banach field over  $k$ . For any  $a \in \mathcal{A}$ ,  $\mathcal{A}\{a\}$  is spectrally reduced. Moreover, if the image of the map  $\mathcal{A}^\times \rightarrow [0, \infty), b \mapsto \|b\|_{\mathcal{A}}\|b^{-1}\|_{\mathcal{A}}$  is bounded, then  $\mathcal{A}\{a\}$  is a Banach field over  $k$  and the image of the map  $\mathcal{A}\{a\}^\times \rightarrow [0, \infty), b \mapsto \|b\|_{\mathcal{A}\{a\}}\|b^{-1}\|_{\mathcal{A}\{a\}}$  is bounded.*

*Proof.* Since  $\mathcal{A}$  is uniform,  $\mathcal{A}$  is spectrally reduced. If  $\|a\|_{\mathcal{A}} \leq 1$ , then  $\mathcal{A}\{a\} \cong \mathcal{A}$ . Therefore we may assume  $\|a\|_{\mathcal{A}} > 1$ . In particular,  $a \neq 0$  and hence  $a \in \mathcal{A}^\times$ . If  $\|a^{-1}\|_{\mathcal{A}} < 1$ , then  $T - a = -a(1 - a^{-1}T) \in \mathcal{A}^\times(1 + \mathcal{A}\{T\}^{\circ\circ}) \subset \mathcal{A}\{T\}^\times$ , and hence  $\mathcal{A}\{a\} = \mathcal{A}\{T\}/\mathcal{A}\{T\}(T - a) \cong 0$ . Therefore we may assume  $\|a^{-1}\|_{\mathcal{A}} \geq 1$ . Let  $n \in \mathcal{A}\{a\}$  be a quasi-nilpotent. We verify  $n = 0$ . Take a lift  $f = \sum_{i=0}^\infty f_i T^i \in \mathcal{A}\{T\}$  of  $n$  with respect to the canonical projection  $\mathcal{A}\{T\} \rightarrow \mathcal{A}\{a\}$ , and assume  $f \neq 0$ . Since  $\|a^{-1}\|_{\mathcal{A}} \geq 1$ ,  $\mathcal{M}(\mathcal{A})\{a\} \neq \emptyset$ . Take an  $x \in \mathcal{M}(\mathcal{A})\{a\} \cong \mathcal{M}(\mathcal{A})\{a\}$ . Since  $n$  is a quasi-nilpotent,  $n(x) = 0 \in \kappa(x)$ , and hence  $f(x) = \sum_{i=0}^\infty f_i(x)T^i \in \kappa(x)\{T\}(T - a(x))$ . The canonical homomorphism  $\mathcal{A} \rightarrow \kappa(x)$  is injective with dense image because  $\mathcal{A}$  is a field. Therefore  $f(x) \neq 0$ . Take a unique  $g_x = \sum_{i=0}^\infty g_{x,i}T^i \in \kappa(x)\{T\}$  with  $(T - a(x))g_x = f(x)$ . We prove that  $g_x$  lies in the image of  $\mathcal{A}\{T\}$ . By the inequality  $(T - a(x))g_x = f(x)$ , we have  $-a(x)g_{x,0} = f_0(x)$  and  $g_{x,i} - a(x)g_{x,i+1} = f_{i+1}(x)$  for any  $i \in \mathbb{N}$ . Since  $|a(x)| = |a^{-1}(x)|^{-1} \geq \|a^{-1}\|_{\mathcal{A}}^{-1} > 0$ ,  $a(x) \neq 0 \in \kappa(x)$  and hence these equalities guarantee that  $g_{x,i}$  lies in the image of  $\mathcal{A}$  for any  $i \in \mathbb{N}$ . Let  $g_i \in \mathcal{A}$  denote a unique element with  $g_i(x) = g_{x,i}$  for each  $i \in \mathbb{N}$ . Set  $g := \sum_{i=0}^\infty g_i T^i \in \mathcal{A}[[T]]$ . Since  $(T - a(x))g(x) = (T - a(x))g_x = f(x) \in \kappa(x)\{T\} \subset \kappa(x)[[T]]$ , we have  $(T - a)g = f \in \mathcal{A}[[T]]$  by the injectivity of  $\mathcal{A} \rightarrow \kappa(x)$ . It implies that  $f_{i+1} = g_i T - a g_{i+1}$  and  $g_i = -\sum_{j=0}^i a^{-j-1} f_{i-j} T^j$  for any  $i \in \mathbb{N}$ . On the other hand,  $T - a = -a(1 - a^{-1}T) \in \mathcal{A}^\times(1 + \mathcal{A}[[T]]T) \subset \mathcal{A}[[T]]^\times$ , and hence  $g = (T - a)^{-1}f \in \mathcal{A}[[T]]$ . Therefore  $g$  is independent of the choice of  $x$ , and satisfies  $(T - a(y))g(y) = f(y) \in \kappa(y)\{T\}$  for any  $y \in \mathcal{M}(\mathcal{A})\{a\}$ . Let  $\epsilon > 0$ . We show that there is an  $N \in \mathbb{N}$  such that  $\|g_i\|_{\mathcal{A}} < 2\epsilon$  for any  $i \geq N$ .

Since  $f \in \mathcal{A}\{T\}$ , there is an  $N_0 \in \mathbb{N}$  such that  $\|f_i\|_{\mathcal{A}} < \epsilon$  for any  $i \geq N_0$ . Assume that there is an  $i \geq N_0$  such that  $\|g_i\|_{\mathcal{A}\{a\}} \geq \epsilon$ . Then by the uniformity of  $\mathcal{A}$ , there is a  $y \in \mathcal{M}(\mathcal{A})\{a\}$  such that  $|g_i(y)| \geq \epsilon$ . By the inequalities  $|f_{i+1}(y)| \leq \|f_{i+1}\|_{\mathcal{A}} < \epsilon$  and  $\|g_i(y)T\|_{\kappa(y)\{T\}} = |g_i(y)| \geq \epsilon$ , we have

$$|g_{i+1}(y)| = |a(y)|^{-1} |(ag_{i+1})(y)| = |a(y)|^{-1} \|g_i(y)T - f_{i+1}(y)\|_{\kappa(y)\{T\}} \geq \epsilon.$$

Therefore  $\overline{\lim}_{n \rightarrow \infty} |g_i(y)| \geq \epsilon$ . It contradicts the fact that  $g(y) \in \kappa(y)\{T\}$ . It implies  $\|g_i\|_{\mathcal{A}\{a\}} < \epsilon$  for any  $i \geq N_0$ .

Let  $z \in \mathcal{M}(\mathcal{A})\{a, a^{-1}\}$ . Since  $g(z) = \sum_{i=0}^\infty g_i(z)T^i \in \kappa(z)\{T\}$ , there is an  $N_1 \in \mathbb{N}$  such that  $|g_i(z)| < \epsilon$  for any  $i \geq N_1$ . Denote by  $N_1(z) \in \mathbb{N}$  the minimum of such an  $N_1$ . Set  $V(z) := \{w \in \mathcal{M}(\mathcal{A}) \mid |f_{N_1(z)}(w)| < \epsilon\}$ . Then  $V(z)$  is an open neighbourhood of  $z$ . For an  $i \geq N_1(z) + N_0$ , suppose  $\sup_{w \in V(z)} |f_i(w)| < \epsilon$ . Then for any  $w \in V(z)$ , by the inequalities  $|f_{i+1}(w)| \leq \|f_{i+1}\|_{\mathcal{A}} < \epsilon$  and  $\|g_i(w)T\|_{\kappa(w)\{T\}} = |g_i(w)| \geq \epsilon$ , we have  $|g_{i+1}(w)| = |a(w)|^{-1} |(ag_{i+1})(w)| = |a(w)|^{-1} \|g_i(w)T - f_{i+1}(w)\|_{\kappa(w)\{T\}} < \epsilon$ . Therefore  $\sup_{w \in V(z)} |g_{i+1}(w)| \leq \epsilon$ . By the induction on  $i$ , we obtain  $\sup_{w \in V(z)} |f_i(w)| < \epsilon$  for any  $i \geq N_1(z) + N_0$ .

Put  $U := \mathcal{M}(\mathcal{A}) \setminus \mathcal{M}(\mathcal{A})\{a\}$ . We have  $U = \bigcup_{r>1} \mathcal{M}(\mathcal{A})\{ra^{-1}\}$ . For any  $r > 1$ ,  $T - a \in \mathcal{A}\{ra^{-1}\}\{T\}^\times$ , and hence the image of  $g$  in  $\mathcal{A}\{ra^{-1}\}[[T]]$  is contained in  $\mathcal{A}\{ra^{-1}\}\{T\}$ .

Consider the open covering

$$\left\{ \bigcup_{r>r_0} \mathcal{M}(\mathcal{A}) \{ra^{-1}\} \mid r_0 > 1 \right\} \sqcup \left\{ V(z) \mid z \in \mathcal{M}(\mathcal{A} \{a, a^{-1}\}) \right\} \\ \sqcup \left\{ \bigcup_{r<1} \mathcal{M}(\mathcal{A}) \{r^{-1}a\} \right\}$$

of  $\mathcal{M}(\mathcal{A})$ . Since  $\mathcal{M}(\mathcal{A})$  is compact, there are an  $n \in \mathbb{N}$ , an  $r_0 > 1$ , and  $z_1, \dots, z_n \in \mathcal{M}(\mathcal{A}) \{a, a^{-1}\}$  such that

$$\mathcal{M}(\mathcal{A}) = \bigcup_{r>r_0} \mathcal{M}(\mathcal{A}) \{ra^{-1}\} \cup \bigcup_{i=1}^n V(z_i) \cup \bigcup_{r<1} \mathcal{M}(\mathcal{A}) \{r^{-1}a\}.$$

Set  $N := \sum_{j=1}^n N_1(z_j) + N_0$ . By the argument above, we obtain  $\|g_i\|_{\mathcal{A}} < 2\epsilon$  for any  $i \geq N$ . It implies  $g \in \mathcal{A} \{T\}$  and  $f \in \mathcal{A} \{T\}(T - a)$ . Thus  $n = f(a) = 0 \in \mathcal{A} \{a\}$ . We conclude that  $\mathcal{A} \{a\}$  is spectrally reduced.

Suppose that the image of the map  $\mathcal{A}^\times \rightarrow [0, \infty), b \mapsto \|b\|_{\mathcal{A}} \|b^{-1}\|_{\mathcal{A}}$  is bounded. Set  $C := \sup_{b \in \mathcal{A}^\times} \|b\|_{\mathcal{A}} \|b^{-1}\|_{\mathcal{A}}$ . Then for any  $b \in \mathcal{A} \{a\}^\times$  lying in the image of  $\mathcal{A}$ , we have  $\|b\|_{\mathcal{A} \{a\}} \|b^{-1}\|_{\mathcal{A} \{a\}} \leq C$  because the canonical homomorphism  $\mathcal{A} \rightarrow \mathcal{A} \{a\}$  is submetric by the definition of  $\|\cdot\|_{\mathcal{A} \{a\}}$ . Let  $b \in \mathcal{A} \{a\} \setminus \{0\}$ . Since the image of  $\mathcal{A}$  is dense in  $\mathcal{A} \{a\}$ , there is a sequence  $(b_i)_{i \in \mathbb{N}} \in \mathcal{A} \{a\}^\times$  such that each  $b_i$  lies in the image of  $\mathcal{A}$  and  $\lim_{i \rightarrow \infty} b_i = b$ . Since  $b \neq 0$  and  $\mathcal{A} \{a\} \setminus \{0\} \subset \mathcal{A} \{a\}$  is open, we may assume  $b_i \neq 0$  for any  $i \in \mathbb{N}$ . Since the underlying  $k$ -algebra of  $\mathcal{A}$  is a field,  $b_i \in \mathcal{A} \{a\}^\times$  for any  $i \in \mathbb{N}$ . Let  $\epsilon > 0$ . Since  $\lim_{i \rightarrow \infty} b_i = b$ , there is an  $N \in \mathbb{N}$  such that  $\|b - b_i\|_{\mathcal{A} \{a\}} < \min\{\|b\|_{\mathcal{A} \{a\}}, \epsilon\}$  for any  $i \geq N$ . Therefore we have

$$\begin{aligned} \|b_i^{-1} - b_j^{-1}\|_{\mathcal{A} \{a\}} &\leq \|b_j - b_i\|_{\mathcal{A} \{a\}} \|b_i^{-1}\|_{\mathcal{A} \{a\}} \|b_j^{-1}\|_{\mathcal{A} \{a\}} \\ &\leq \|b_j - b_i\|_{\mathcal{A} \{a\}} \left( C \|b_i\|_{\mathcal{A} \{a\}}^{-1} \right) \left( C \|b_j\|_{\mathcal{A} \{a\}}^{-1} \right) < C^2 \|b\|_{\mathcal{A} \{a\}}^{-2} \epsilon \end{aligned}$$

for any  $i, j \geq N$ . It implies that  $(b_i^{-1})_{i \in \mathbb{N}}$  converges, and the limit is the inverse of  $b$  by the continuity of the multiplication. In particular,  $b \in \mathcal{A} \{a\}^\times$ . Therefore  $\mathcal{A} \{a\}$  is a Banach field over  $k$ . The inverse  $(\cdot)^{-1} : \mathcal{A} \{a\}^\times \rightarrow \mathcal{A} \{a\}^\times$  is continuous by [BGR] 1.2.4. Proposition 4, and hence the map  $\mathcal{A} \{a\}^\times \rightarrow [0, \infty), b \mapsto \|b\|_{\mathcal{A} \{a\}} \|b^{-1}\|_{\mathcal{A} \{a\}}$  is continuous. Since the image of  $\mathcal{A}$  is dense in  $\mathcal{A} \{a\}$ , we obtain  $\sup_{b \in \mathcal{A} \{a\}^\times} \|b\|_{\mathcal{A} \{a\}} \|b^{-1}\|_{\mathcal{A} \{a\}} \leq C$ .  $\square$

Let  $\mathcal{A}$  be a Banach  $k$ -algebra or  $\mathbb{R}$ . For a topological space  $X$ , we denote by  $C_{\text{bd}}(X, \mathcal{A})$  the algebra of bounded continuous  $\mathcal{A}$ -valued functions on  $X$  endowed with the supremum norm  $\|\cdot\|_{C_{\text{bd}}(X, \mathcal{A})}$  on  $X$ . Suppose that  $\mathcal{A}$  is a uniform Banach  $k$ -algebra. Then  $C_{\text{bd}}(X, \mathcal{A})$  is a uniform Banach  $k$ -algebra. If  $X$  is compact, then  $C_{\text{bd}}(X, \mathcal{A})$  coincides with the Banach  $K$ -algebra of continuous  $\mathcal{A}$ -valued functions on  $X$ . We mainly consider the case  $\mathcal{A} = k$ . We remark that a rational domain of  $\mathcal{M}(C_{\text{bd}}(X, k))$  does not necessarily correspond to

a clopen subset of  $X$ . It is because there are monstrously many points in  $\mathcal{M}(\mathbf{C}_{\text{bd}}(X, k))$  which are not  $k$ -rational when  $X$  is not compact and  $k$  is not a local field. For example, there are monstrously many rational domains which do not intersect with the image of  $X$ . For more detail, see [Mih].

**Definition 2.9.** A Banach  $k$ -algebra  $\mathcal{A}$  is said to be spectral if  $\mathcal{M}(\mathcal{A})(k) \neq \emptyset$  and the equality  $\|a\|_{\mathcal{A}} = \sup_{x \in \mathcal{M}(\mathcal{A})(k)} |a(x)|$  holds for any  $a \in \mathcal{A}$ .

Every spectral Banach  $k$ -algebra is uniform, and is canonically isometrically isomorphic to a closed  $k$ -subalgebra of the Banach  $k$ -algebra  $\mathbf{C}_{\text{bd}}(\mathcal{M}(\mathcal{A})(k), k)$  through the Gel'fand transform on  $\mathcal{M}(\mathcal{A})(k)$ .

**Theorem 2.10.** If  $k$  is a local field, then every spectral  $k$ -algebra is sheafy.

*Proof.* Let  $\mathcal{A}$  be a spectral  $k$ -algebra. The image of the Gel'fand transform  $\mathcal{A} \hookrightarrow \mathbf{C}_{\text{bd}}(\mathcal{M}(\mathcal{A})(k), k)$  separates points. By [Mih] Theorem 5.8, there is a one-to-one correspondence between closed  $k$ -subalgebras of  $\mathbf{C}_{\text{bd}}(\mathcal{M}(\mathcal{A})(k), k)$  separating points and totally disconnected Hausdorff quotient spaces of  $\mathcal{M}(\mathbf{C}_{\text{bd}}(\mathcal{M}(\mathcal{A})(k), k))$  which is faithful under  $\mathcal{M}(\mathcal{A})(k)$ . In particular,  $\mathcal{A}$  is canonically isometrically isomorphic to  $\mathbf{C}_{\text{bd}}(X, k)$  for a totally disconnected compact Hausdorff space  $X$ . Therefore we may assume  $\mathcal{A} = \mathbf{C}_{\text{bd}}(\mathcal{M}(\mathcal{A})(k), k)$  for a compact space  $X$  without loss of generality. Each  $x \in X$  corresponds to a point of  $\mathcal{M}(\mathbf{C}_{\text{bd}}(X, k))$  by the evaluation at  $x$ . This correspondence gives a continuous map  $\iota_X: X \rightarrow \mathcal{M}(\mathbf{C}_{\text{bd}}(X, k))$ . Since  $X$  is a totally disconnected compact Hausdorff space, the natural map  $X \rightarrow \mathcal{M}(\mathbf{C}_{\text{bd}}(X, k))$  is a homeomorphism by Gel'fand–Naimark theorem ([Ber1] 9.2.5. Theorem (i)).

Let  $U \subset \mathcal{M}(\mathbf{C}_{\text{bd}}(X, k))$  be a rational domain. Take generators  $f_0, f_1, \dots, f_n$  of the principal ideal (1) with  $U = \{x \in \mathcal{M}(\mathbf{C}_{\text{bd}}(X, k)) \mid |f_i(x)| \leq |f_0(x)|, \forall i \in \mathbb{N} \cap [1, n]\}$ . Since  $k$  is zero-dimensional, the subset  $U \cap X := \{x \in X \mid |f_i(x)| \leq |f_0(x)|, \forall i \in \mathbb{N} \cap [1, n]\}$  is clopen. Therefore  $U \subset \mathcal{M}(\mathbf{C}_{\text{bd}}(X, k))$  is clopen because it is the image of  $U \cap X$  by  $\iota_X$ . Let  $e \in \mathbf{C}_{\text{bd}}(X, k)$  be the characteristic function of  $U \cap X$ . Since  $1 - e$  vanishes at every point of  $U$ ,  $1 - e$  is contained in the kernel of the canonical bounded homomorphism  $\mathbf{C}_{\text{bd}}(X, k) \rightarrow \mathbf{C}_{\text{bd}}(X, k)\{f_0^{-1}f_1, \dots, f_0^{-1}f_n\}$ . Let  $f \in \mathbf{C}_{\text{bd}}(X, k)$ . Since  $\iota_X|_{U \cap X}: U \cap X \rightarrow U$  is surjective, we have  $\|f_U\|_{\mathbf{C}_{\text{bd}}(X, k)\{f_0^{-1}f_1, \dots, f_0^{-1}f_n\}} \geq \|f|_{U \cap X}\|_{\mathbf{C}_{\text{bd}}(U \cap X, k)}$ . On the other hand, the equality  $\|(1 - e)f\|_{\mathcal{M}(\mathbf{C}_{\text{bd}}(X, k))} = \|f|_{U \cap X}\|_{\mathcal{M}(\mathbf{C}_{\text{bd}}(U \cap X, k))}$  implies  $\|f_U\|_{\mathbf{C}_{\text{bd}}(X, k)\{f_0^{-1}f_1, \dots, f_0^{-1}f_n\}} \leq \|f|_{U \cap X}\|_{\mathbf{C}_{\text{bd}}(U \cap X, k)}$ . Thus  $\|f_U\|_{\mathbf{C}_{\text{bd}}(X, k)\{f_0^{-1}f_1, \dots, f_0^{-1}f_n\}} = \|f|_{U \cap X}\|_{\mathbf{C}_{\text{bd}}(U \cap X, k)}$ . Since  $U \cap X \subset X$  is clopen, the restriction  $(\cdot)|_{U \cap X}: \mathbf{C}_{\text{bd}}(X, k) \rightarrow \mathbf{C}_{\text{bd}}(U \cap X, k)$  is surjective. Therefore by the argument above,  $\mathbf{C}_{\text{bd}}(X, k)\{f_0^{-1}f_1, \dots, f_0^{-1}f_n\}$  and  $\mathbf{C}_{\text{bd}}(U \cap X, k)$  are isometrically isomorphic to each other. In particular, the isomorphism class of  $\mathbf{C}_{\text{bd}}(X, k)\{f_0^{-1}f_1, \dots, f_0^{-1}f_n\}$  is independent of the presentation  $(n, (f_0, f_1, \dots, f_n))$  of  $U$ , and the structure presheaf  $\mathcal{O}_{\mathbf{C}_{\text{bd}}(X, k)}$  on the Grothendieck topology generated by rational domains on  $\mathcal{M}(\mathbf{C}_{\text{bd}}(X, k))$  is well-defined. Moreover,  $\mathcal{O}_{\mathbf{C}_{\text{bd}}(X, k)}$  is canonically isomorphic to the sheaf of bounded continuous functions. We conclude that  $\mathcal{A} = \mathbf{C}_{\text{bd}}(X, k)$  is sheafy.  $\square$

### 3 Uniformity of Adic Spectra

In this paper, an affinoid ring  $\mathcal{A} = (\mathcal{A}^\flat, \mathcal{A}^+)$  is assumed to satisfy that  $\mathcal{A}^\flat$  is an f-adic Tate ring. We observe the relation between properties of an affinoid ring and norms on the underlying topological ring in §3.1. Using the results, we verify that the example in §2.2 yields examples of uniform affinoid rings whose rational localisations are not uniform in §3.2. We verify that one of them is an example of a non-sheafy uniform affinoid ring.

#### 3.1 Norms on Affinoid Rings

**Definition 3.1.** Let  $\mathcal{A}$  be a Banach  $k$ -algebra. We denote by  $\mathcal{A}^{\text{ad}+} \subset \mathcal{A}$  the integral closure of  $\mathcal{A}(1)$ , and call  $\mathcal{A}^{\text{ad}} := (\mathcal{A}, \mathcal{A}^{\text{ad}+})$  the affinoid algebra associated to  $\mathcal{A}$ .

We remark that every Banach  $k$ -algebra  $\mathcal{A}$  is a complete f-adic Tate ring admitting a ring of definition which is a  $k(1)$ -subalgebra, because  $k$  admits a topologically nilpotent unit and  $\mathcal{A}(1)$  is a ring of definition. Therefore  $\mathcal{A}^{\text{ad}}$  is an affinoid ring over  $k^{\text{ad}}$ . In addition if  $\mathcal{A}$  is a uniform Banach  $k$ -algebra, then  $\mathcal{A}(1) = \mathcal{A}^\circ$  is integrally closed and hence  $\mathcal{A}^{\text{ad}} = (\mathcal{A}, \mathcal{A}(1)) = (\mathcal{A}, \mathcal{A}^\circ)$ .

**Definition 3.2.** An affinoid ring  $\mathcal{A}$  over  $k^{\text{ad}}$  is said to be Banach if  $\mathcal{A}^\flat$  is a complete f-adic Tate ring admitting a ring of definition which is a  $k(1)$ -subalgebra.

**Proposition 3.3.** An affinoid ring  $\mathcal{A}$  over  $k^{\text{ad}}$  is Banach if and only if  $\mathcal{A}^\flat$  admits a norm as a  $k$ -algebra which gives the original topology of it.

*Proof.* It suffices to verify direct implication by the argument right after Definition 3.1. Suppose  $\mathcal{A}^\flat$  is a complete f-adic Tate ring admitting a ring of definition  $\mathcal{A}_0$  which is a  $k(1)$ -subalgebra. For any  $\epsilon \in k(1-)$ , we have  $\epsilon \mathcal{A}_0^{\text{ac}} \subset \mathcal{A}_0 \subset \mathcal{A}_0^{\text{ac}}$  because the valuation of  $k$  is non-trivial, and hence  $\mathcal{A}_0^{\text{ac}}$  is also a ring of definition which is a  $k(1)$ -subalgebra. Therefore the topology of  $\mathcal{A}^\flat$  is given by  $\|\cdot\|_{\mathcal{A}^\flat, \mathcal{A}_0}$  because  $\mathcal{A}_0^{\text{ac}}$  is the closed unit ball of  $\mathcal{A}^\flat$  with respect to  $\|\cdot\|_{\mathcal{A}^\flat, \mathcal{A}_0}$ . Moreover,  $\|\cdot\|_{\mathcal{A}^\flat, \mathcal{A}_0}$  is a norm of  $\mathcal{A}$  as a  $k$ -algebra by Lemma 1.10.  $\square$

We note that Proposition 3.3 does not guarantee that every Banach affinoid ring over  $k^{\text{ad}}$  is isomorphic to the affinoid algebra associated to a Banach  $k$ -algebra. For any Banach  $k$ -algebra  $\mathcal{A}$ ,  $\mathcal{A}(1)$  is a ring of definition. On the other hand, for a general Banach affinoid ring  $\mathcal{A}$  over  $k^{\text{ad}}$ , we do not know whether there is a ring of definition  $\mathcal{A}_0 \subset \mathcal{A}^\flat$  such that  $\mathcal{A}^+$  coincides with the integral closure of  $\mathcal{A}_0$ . Such a problem does not occur when we are dealing with a Banach affinoid ring  $\mathcal{A}$  over  $k^{\text{ad}}$  such that  $\mathcal{A}^+$  is bounded.

**Definition 3.4.** An affinoid ring  $\mathcal{A}$  over  $k^{\text{ad}}$  is said to be uniform if  $\mathcal{A}$  is Banach and  $(\mathcal{A}^\flat)^\circ$  is bounded, and is said to be strongly uniform if  $\mathcal{A}$  is uniform and  $\mathcal{A}^+ = (\mathcal{A}^\flat)^\circ$ .

In particular, for any uniform affinoid ring  $\mathcal{A}$  over  $k^{\text{ad}}$ ,  $\mathcal{A}^+ \subset (\mathcal{A}^\flat)^\circ$  is also bounded. Therefore every uniform affinoid ring over  $k^{\text{ad}}$  is obtained as the affinoid algebra associated to a Banach  $k$ -algebra. Moreover, a stronger fact holds as is shown in the following proposition.

**Proposition 3.5.** *An affinoid ring  $\mathcal{A}$  over  $k^{\text{ad}}$  is uniform if and only if  $\mathcal{A}$  is isomorphic to the affinoid algebra associated to a Banach function algebra over  $k$ .*

*Proof.* It immediately follows from Proposition 1.11 and Proposition 3.3.  $\square$

**Corollary 3.6.** *For any uniform affinoid ring  $\mathcal{A}$  over  $k^{\text{ad}}$ , there is a unique norm  $\|\cdot\|_{\mathcal{A}}$  on  $\mathcal{A}^{\circ}$  such that  $\|\cdot\|_{\mathcal{A}}$  gives the original topology of  $\mathcal{A}^{\circ}$  and the closed unit ball of  $\mathcal{A}^{\circ}$  with respect to  $\|\cdot\|_{\mathcal{A}}$  coincides with  $\mathcal{A}^{\circ}$ . Moreover,  $\|\cdot\|_{\mathcal{A}}$  is complete and power-multiplicative.*

**Corollary 3.7.** *For any uniform affinoid ring  $\mathcal{A}$  over  $k^{\text{ad}}$ ,  $(\mathcal{A}^{\circ})^{\circ}$  is adically closed.*

*Proof.* It immediately follows from Corollary 1.12 and Proposition 3.5.  $\square$

**Proposition 3.8.** *An affinoid ring  $\mathcal{A}$  over  $k^{\text{ad}}$  is strongly uniform if and only if  $\mathcal{A}$  is isomorphic to the affinoid algebra associated to a uniform Banach  $k$ -algebra.*

*Proof.* It immediately follows from Corollary 3.6 and Corollary 3.7.  $\square$

**Corollary 3.9.** *For any strongly uniform affinoid ring  $\mathcal{A}$  over  $k^{\text{ad}}$ , there is a unique norm  $\|\cdot\|_{\mathcal{A}}$  on  $\mathcal{A}^{\circ}$  such that  $\|\cdot\|_{\mathcal{A}}$  gives the original topology of  $\mathcal{A}^{\circ}$  and the closed unit ball of  $\mathcal{A}^{\circ}$  with respect to  $\|\cdot\|_{\mathcal{A}}$  coincides with  $\mathcal{A}^+$ . Moreover,  $\|\cdot\|_{\mathcal{A}}$  is complete and power-multiplicative.*

**Proposition 3.10.** *Suppose that  $|k|$  is dense in  $[0, \infty)$ . An affinoid ring  $\mathcal{A}$  over  $k^{\text{ad}}$  is uniform if and only if  $(\mathcal{A}^{\circ}, (\mathcal{A}^+)^{\text{ac}})$  is strongly uniform.*

*Proof.* Since  $(\mathcal{A}^+)^{\text{ac}} \subset \mathcal{A}^{\circ}$  is an open integrally closed  $k(1)$ -subalgebra by Proposition 1.9,  $(\mathcal{A}^{\circ}, (\mathcal{A}^+)^{\text{ac}})$  is an affinoid ring over  $k^{\text{ad}}$ . The direct implication follows from Corollary 3.6. Since  $\mathcal{A}$  and  $(\mathcal{A}^{\circ}, (\mathcal{A}^+)^{\text{ac}})$  share the underlying topological  $k$ -algebra, one of them is Banach if and only if the other of it is Banach. Therefore we may assume  $\mathcal{A}^{\circ}$  admits a norm which gives the topology of it by Proposition 3.3. First, suppose that  $\mathcal{A}$  is uniform. Then  $(\mathcal{A}^{\circ})^{\circ}$  is adically closed by Corollary 3.7. We obtain  $(\mathcal{A}^+)^{\text{ac}} \subset ((\mathcal{A}^{\circ})^{\circ})^{\text{ac}} = (\mathcal{A}^{\circ})^{\circ}$ . Let  $a \in (\mathcal{A}^{\circ})^{\circ}$ . Then for any  $\epsilon \in k(1-)$ ,  $\epsilon a$  is topologically nilpotent. Therefore there is an  $n \in \mathbb{N} \setminus \{0\}$  such that  $(\epsilon a)^n \in \mathcal{A}^+$  because  $\mathcal{A}^+$  is an open neighbourhood of 0. Since  $\mathcal{A}^+$  is integrally closed, we get  $\epsilon a \in \mathcal{A}^+$ . It implies  $a \in (\mathcal{A}^+)^{\text{ac}}$ . Thus  $\mathcal{A}^{\circ} = (\mathcal{A}^+)^{\text{ac}}$ , and hence  $(\mathcal{A}^{\circ}, (\mathcal{A}^+)^{\text{ac}})$  is strongly uniform by Proposition 3.8. Secondly, suppose that  $(\mathcal{A}^{\circ}, (\mathcal{A}^+)^{\text{ac}})$  is strongly uniform. Then  $(\mathcal{A}^{\circ})^{\circ} = (\mathcal{A}^+)^{\text{ac}}$  is bounded by Proposition 3.8, and hence  $\mathcal{A}$  is uniform by Proposition 3.5.  $\square$

## 3.2 Negative Facts

We give an example of strongly uniform affinoid ring such that its rational localisation is not uniform. The existence of such an example implies that the uniformisation of the affinoid algebra associated to a Banach  $k$ -algebra given in [Ber1] 1.3. (the strong uniformisation of a uniform Banach affinoid ring given in Proposition 3.10) does not induce a uniformisation (strong uniformisation) of the adic spectrum.

**Theorem 3.11.** *Let  $r \in (1, \infty)$ . Set  $Y := \text{Spa}(k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}^{\text{ad}})$  and  $V := \{x \in Y \mid |X(x)| \leq 1\}$ . The rational localisation  $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$  is not a uniform affinoid ring over  $k^{\text{ad}}$ .*

*Proof.* By Proposition 2.1,  $\mathcal{O}_Y(V)$  is isomorphic to the underlying topological  $k$ -algebra of  $k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{X\}$ . Therefore  $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$  is not uniform by Theorem 2.4 and Proposition 3.5, because any complete norm on the underlying topological  $k$ -algebra of a Banach  $k$ -algebra which gives the original topology of it is equivalent to the original norm by Banach open mapping theorem ([Bou], I.3.3. Theorem 1).  $\square$

We remark that the direct analogue of Corollary 2.5 does not hold for the adic spectrum because the structure presheaf on an adic spectrum is always well-defined by the universality of rational domains. Indeed, the rational domain

$$V' := \{x \in Y \mid |X(x)| \leq 1, |(UX)(x)| \leq 1\}$$

does not coincide with  $V$  unlike the corresponding rational domains of the Berkovich spectrum, because  $UX \in (\mathcal{O}_Y(V)^\circ)^{\text{ac}} \setminus \mathcal{O}_Y(V)^\circ$ . They share points of height 1, but  $V$  possesses more points of general height than  $V'$ . This example seems not to be so pathological, and hence we are not sure whether it is sheafy or not. Using it, we construct a more pathological example.

For a Banach  $k$ -vector space  $V$ , we denote by  $|V|$  the underlying  $k$ -vector space of  $V$ . For a family  $(V_i)_{i \in I}$  of Banach  $k$ -vector spaces indexed by a set  $I$ , we set

$$\prod_{i \in I} V_i := \left\{ v = (v_i)_{i \in I} \in \prod_{i \in I} |V_i| \mid \sup_{i \in I} \|v_i\|_{V_i} < \infty \right\}$$

and endow it with the norm  $\|\cdot\|_{\prod_{i \in I} V_i} : v = (v_i)_{i \in I} \mapsto \sup_{i \in I} \|v_i\|_{V_i}$ . Then  $(\prod_{i \in I} V_i, \|\cdot\|_{\prod_{i \in I} V_i})$  is a Banach  $k$ -vector space, and satisfies the universality of a direct product of  $(V_i)_{i \in I}$  in the category of Banach  $k$ -vector spaces and submetric  $k$ -linear homomorphisms. In addition if  $\#I < \infty$ , it satisfies the same universality in the category of Banach  $k$ -vector spaces and bounded  $k$ -linear homomorphisms. If  $V_i$  is the underlying Banach  $k$ -vector space of Banach  $k$ -algebra  $\mathcal{A}_i$  for each  $i \in I$ , then the multiplication of  $\mathcal{A}_i$  for each  $i \in I$  induces a  $k$ -algebra structure of  $\prod_{i \in I} V_i$ , and we denote by  $\prod_{i \in I} \mathcal{A}_i$  the resulting  $k$ -algebra endowed with the norm  $\|\cdot\|_{\prod_{i \in I} \mathcal{A}_i} := \|\cdot\|_{\prod_{i \in I} V_i}$ . Then  $(\prod_{i \in I} \mathcal{A}_i, \|\cdot\|_{\prod_{i \in I} \mathcal{A}_i})$  is a Banach  $k$ -algebra, and satisfies the universality of a direct product of  $(\mathcal{A}_i)_{i \in I}$  in the category of Banach  $k$ -algebras and submetric  $k$ -algebra homomorphisms. Moreover,  $\prod_{i \in I} \mathcal{A}_i$  is uniform if and only if so is  $\mathcal{A}_i$  for any  $i \in I$ . We remark that a direct product does not respect the class of Banach function algebras over  $k$ , but at least it is true that if  $\prod_{i \in I} \mathcal{A}_i$  is a Banach function algebra over  $k$ , then so is  $\mathcal{A}_i$  because the zero-extension  $\mathcal{A}_i \hookrightarrow \prod_{i \in I} \mathcal{A}_i$  is an isometric multiplicative  $k$ -linear homomorphism for any  $i \in I$ . There is a natural isometric homomorphism

$$\left( \prod_{i \in I} \mathcal{A}_i \right) \{T\} \hookrightarrow \prod_{i \in I} (\mathcal{A}_i \{T\})$$

$$\sum_{h=0}^{\infty} (a_{h,i})_{i \in I} T^h \mapsto \left( \sum_{h=0}^{\infty} a_{h,i} T^h \right)_{i \in I},$$

which is not surjective unless  $\mathcal{A}_i \cong 0$  for all but finitely many  $i \in I$ . We identify the domain with the image. For an  $(a_i)_{i \in I} \in \prod_{i \in I} \mathcal{A}_i$ , it induces a bounded homomorphism

$$\left( \prod_{i \in I} \mathcal{A}_i \right) \{(a_i)_{i \in I}\} \hookrightarrow \prod_{i \in I} (\mathcal{A}_i \{a_i\}),$$

which is submetric by the definition of the quotient norms.

**Lemma 3.12.** *Let  $r \in C_{\text{bd}}(\mathbb{N}, \mathbb{R})$  with  $r(\mathbb{N}) \subset (1, \infty)$ . Then the canonical homomorphism*

$$\left( \prod_{n \in \mathbb{N}} k \left\{ r(n)^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N} \right\} \right) \{(X)_{n \in \mathbb{N}}\} \rightarrow \prod_{n \in \mathbb{N}} \left( k \left\{ r(n)^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N} \right\} \{X\} \right)$$

is an isometry.

*Proof.* For each  $n \in \mathbb{N}$ , put

$$\mathcal{A}_n := k \left\{ r(n)^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N} \right\}$$

Denote by  $\varphi: (\prod_{n \in \mathbb{N}} \mathcal{A}_n) \{(X)_{n \in \mathbb{N}}\} \rightarrow \prod_{n \in \mathbb{N}} (\mathcal{A}_n \{X\})$  the canonical homomorphism. Let  $f \in (\prod_{n \in \mathbb{N}} \mathcal{A}_n) \{(X)_{n \in \mathbb{N}}\}$ . Take a lift

$$F = \sum_{h=0}^{\infty} F_h T^h = \sum_{h=0}^{\infty} (F_{h,n}(U, X))_{n \in \mathbb{N}} T^h \in \left( \prod_{n \in \mathbb{N}} \mathcal{A}_n \right) \{T\}$$

of  $f$ . Let  $R > \|\varphi(f)\|_{\prod_{n \in \mathbb{N}} (\mathcal{A}_n \{X\})}$ . By the definition of the quotient norm, there is a

$$G = \left( \sum_{h=0}^{\infty} G_{h,n}(U, X) T^h \right)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} (\mathcal{A}_n \{T\})$$

such that  $\|F - ((T)_{n \in \mathbb{N}} - (X)_{n \in \mathbb{N}})G\|_{\prod_{n \in \mathbb{N}} (\mathcal{A}_n \{T\})} < R$ . It implies that  $\|F_{0,n} + XG_{0,n}\|_{\mathcal{A}_n} < R$  and  $\|F_{h+1,n} - G_{h,n} + XG_{h+1,n}\|_{\mathcal{A}_n} < R$  for any  $(h, n) \in \mathbb{N} \times \mathbb{N}$ . Put

$$\begin{aligned} F_{h,n} &= \sum_{j=0}^{\infty} F_{h,n,j}(U) X^j \\ G_{h,n} &= \sum_{j=0}^{\infty} G_{h,n,j}(U) X^j \end{aligned}$$

for each  $(h, n) \in \mathbb{N} \times \mathbb{N}$ . Let  $N \in \mathbb{N}$  denote the smallest integer such that

$$\|F_h\|_{\prod_{n \in \mathbb{N}} \mathcal{A}_n} < \|r\|_{C_{\text{bd}}(\mathbb{N}, \mathbb{R})}^{-1} R$$



$$\|F_h - G_{h-1} + XG_h\|_{\prod_{n \in \mathbb{N}} \mathcal{A}_n} < \|r\|_{\text{Cbd}(\mathbb{N}, \mathbb{R})}^{-1} R$$

for any  $h > N$ . Then for any  $(h, n, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$  with  $h \geq N$ , we have

$$\begin{aligned} & \|G_{h,n,j+1}X^{j+1} - G_{h+1,n,j}X^j\|_{\mathcal{A}_n} \leq \|G_{h,n} - XG_{h+1,n}\|_{\mathcal{A}_n} \\ & = \|F_{h+1,n} - (F_{h+1,n} - G_{h,n} + XG_{h+1,n})\|_{\mathcal{A}_n} < \|r\|_{\text{Cbd}(\mathbb{N}, \mathbb{R})}^{-1} R. \end{aligned}$$

Therefore for any  $(h, n, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$  with  $h > N$ , putting  $m := j + h - N - 1$ , we obtain

$$\begin{aligned} \|G_{h,n,j}X^j\|_{\mathcal{A}_n} & = \left\| G_{N+1,n,m}X^m + \sum_{l=0}^{m-j} (G_{N+l+1,n,m-l}X^{m-l} - G_{N+l,n,m-l+1}X^{m-l+1}) \right\|_{\mathcal{A}_n} \\ & < \|r\|_{\text{Cbd}(\mathbb{N}, \mathbb{R})}^{-1} R, \end{aligned}$$

and it implies  $\|G_{h,n}\|_{\mathcal{A}_n} < \|r\|_{\text{Cbd}(\mathbb{N}, \mathbb{R})}^{-1} R$ . Put

$$H = \sum_{h=0}^N (G_{h,n})_{n \in \mathbb{N}} T^h \in \left( \prod_{n \in \mathbb{N}} \mathcal{A}_n \right) \{T\}.$$

By the argument above, we get  $\|G - H\|_{\prod_{n \in \mathbb{N}} (\mathcal{A}_n \{T\})} < \|r\|_{\text{Cbd}(\mathbb{N}, \mathbb{R})}^{-1} R$ , and hence

$$\begin{aligned} & \|F - (T - (X)_{n \in \mathbb{N}})H\|_{(\prod_{n \in \mathbb{N}} \mathcal{A}_n) \{T\}} = \|F - ((T)_{n \in \mathbb{N}} - (X)_{n \in \mathbb{N}})H\|_{\prod_{n \in \mathbb{N}} (\mathcal{A}_n \{T\})} \\ & = \|(F - ((T)_{n \in \mathbb{N}} - (X)_{n \in \mathbb{N}})G) + ((T)_{n \in \mathbb{N}} - (X)_{n \in \mathbb{N}})(G - H)\|_{\prod_{n \in \mathbb{N}} (\mathcal{A}_n \{T\})} \\ & < \max \left\{ R, \|((T)_{n \in \mathbb{N}} - (X)_{n \in \mathbb{N}})\|_{\prod_{n \in \mathbb{N}} (\mathcal{A}_n \{T\})} \|r\|_{\text{Cbd}(\mathbb{N}, \mathbb{R})}^{-1} R \right\} \\ & = \max \left\{ R, \left( \sup_{n \in \mathbb{N}} r(n) \right) \|r\|_{\text{Cbd}(\mathbb{N}, \mathbb{R})}^{-1} R \right\} = R. \end{aligned}$$

Thus  $\|f\|_{(\prod_{n \in \mathbb{N}} \mathcal{A}_n) \{(X)_{n \in \mathbb{N}}\}} = \|F + (T - (X)_{n \in \mathbb{N}})\|_{(\prod_{n \in \mathbb{N}} \mathcal{A}_n) \{T\}} < R$ , and it implies

$$\|f\|_{(\prod_{n \in \mathbb{N}} \mathcal{A}_n) \{(X)_{n \in \mathbb{N}}\}} = \|\varphi(f)\|_{\prod_{n \in \mathbb{N}} (\mathcal{A}_n \{X\})}$$

because  $\varphi$  is submetric. We conclude that  $\varphi$  is an isometry.  $\square$

**Theorem 3.13.** *Let  $r \in \text{Cbd}(\mathbb{N}, \mathbb{R})$  with  $r(\mathbb{N}) \subset (1, \infty)$ . Set*

$$\begin{aligned} Y &:= \text{Spa} \left( \left( \prod_{n \in \mathbb{N}} k \left\{ r(n)^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N} \right\} \right)^{\text{ad}} \right) \\ V &:= \{ x \in Y \mid |(X)_{n \in \mathbb{N}}(x)| \leq 1 \}. \end{aligned}$$

*The rational localisation  $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$  is not a uniform affinoid ring over  $k^{\text{ad}}$ .*

*Proof.* It immediately follows from Proposition 2.1, Corollary 3.7, and Lemma 3.12 because the proof of Theorem 3.11 shows  $(UX)_{n \in \mathbb{N}} \in (\mathcal{O}_Y(V)^\circ)^{\text{ac}} \setminus \mathcal{O}_Y(V)^\circ$ .  $\square$

This new example is much more significant than  $k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{X\}$  for study of Tate acyclicity. For any  $f \in k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{X\}$ , the multiplication of  $T - f \in k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{X\}\{T\}$  seems to be admissible. Indeed,  $k\{r(n)^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}\{X\}$  is isometrically embedded in the completion  $K\{r^{-b_i}U^i \mid i \in \mathbb{N}\} \subset K[[U]]$  of  $K[U]$  with respect to the norm

$$\|\cdot\|_{K\{r^{-b_i}U^i \mid i \in \mathbb{N}\}} : K[U] \rightarrow [0, \infty)$$

$$\sum_{i=0}^{\infty} F_i U^i \mapsto \sup_{i \in \mathbb{N}} |F_i| r^{b_i}$$

by the proof of Theorem 3.11, where  $K/k$  is the extension of complete valuation fields obtained as the completion of the fractional field of  $k\{X\}$  with respect to the Gauss norm of radius 1. Therefore the computation of the admissibility is reduced to that for  $K\{r^{-b_i}U^i \mid i \in \mathbb{N}\}\{T\}$ , which is not so pathologic. Such an admissibility works well in the calculation of the Čech-complex for Tate acyclicity, and hence  $k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}$  could be still sheafy. On the other hand, the same does not hold for the new example as is shown in the following.

**Proposition 3.14.** *Suppose that  $|k|$  is dense in  $[0, \infty)$ . Let  $\pi \in C_{\text{bd}}(\mathbb{N}, k)$  be a sequence such that  $(|\pi(n)|)_{n \in \mathbb{N}} \in C_{\text{bd}}(\mathbb{N}, \mathbb{R})$  is an increasing sequence converging to 1. Then for any  $r \in (1, \infty)$ , the multiplication of  $T - (\pi(n)UX)_{n \in \mathbb{N}}$  is not an admissible epimorphism on  $(\prod_{n \in \mathbb{N}} k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\})\{(X)_{n \in \mathbb{N}}\}\{T\}$ .*

*Proof.* Put  $\mathcal{A} := k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\}$ . By Lemma 3.12 and the proof of Theorem 2.4, the underlying  $k$ -algebra of  $(\prod_{n \in \mathbb{N}} \mathcal{A})\{(X)_{n \in \mathbb{N}}\}$  is embedded in the underlying  $k$ -algebra of  $\prod_{n \in \mathbb{N}} k[[X, U]]$ , and hence  $(\pi(n)UX)_{n \in \mathbb{N}}$  is not a zero divisor of  $(\prod_{n \in \mathbb{N}} \mathcal{A})\{(X)_{n \in \mathbb{N}}\}$ . Let  $R > 0$ . Since  $(|\pi(n)|)_{n \in \mathbb{N}}$  converges to 1, there is an  $(i_0, n_0) \in \mathbb{N} \times \mathbb{N}$  such that  $|\pi(n_0)|^{i_0} r^{b_{i_0}} > R^{-1}$ . Since  $\rho_{\mathcal{A}}(\pi(n_0)UX) = |\pi(n_0)| < 1$  by the proof of Theorem 2.4,  $\pi(n_0)UX$  is topologically nilpotent and hence there is an  $N > i_0$  such that  $\|(\pi(n_0)UX)^N\| \leq 1$ . Denote by  $e_{n_0} \in (\prod_{n \in \mathbb{N}} \mathcal{A})\{(X)_{n \in \mathbb{N}}\}$  the image of  $1 \in \mathcal{A}\{X\}$  by the zero-extension  $\mathcal{A}\{X\} \hookrightarrow \prod_{n \in \mathbb{N}} (\mathcal{A}\{X\})$  outside the  $n_0$ -th entry. It lies in the image of  $(\prod_{n \in \mathbb{N}} \mathcal{A})\{X\}$ . Then we have

$$\left\| \sum_{i=0}^{N-1} (\pi(n_0)UX)^{N-1-i} e_{n_0} T^i \right\|_{(\prod_{n \in \mathbb{N}} \mathcal{A})\{X\}\{T\}} \geq \|(\pi(n_0)UX)^{i_0}\|_{\mathcal{A}\{X\}} > R^{-1}$$

and

$$\begin{aligned} & \left\| (T - (\pi(n)UX)_{n \in \mathbb{N}}) \sum_{i=0}^{N-1} (\pi(n_0)UX)^{N-1-i} e_{n_0} T^i \right\|_{(\prod_{n \in \mathbb{N}} \mathcal{A})\{X\}\{T\}} \\ &= \|e_{n_0} T^N - (\pi(n_0)UX)^N e_{n_0}\|_{(\prod_{n \in \mathbb{N}} \mathcal{A})\{X\}\{T\}} = 1. \end{aligned}$$

It implies that the inverse  $(T - (\pi(n)UX)_{n \in \mathbb{N}})^{-1}$  of the bounded bijective  $k$ -linear homomorphism  $(\prod_{n \in \mathbb{N}} k\{r^{-a_i}U^iX^{a_i} \mid i \in \mathbb{N}\})\{(X)_{n \in \mathbb{N}}\}\{T\} \rightarrow (T - (\pi(n)UX)_{n \in \mathbb{N}})$  given by the multiplication of  $T - (\pi(n)UX)_{n \in \mathbb{N}}$  has operator norm greater than  $R$ . Thus  $(T - (\pi(n)UX)_{n \in \mathbb{N}})^{-1}$  is not bounded, and hence the multiplication of  $T - (\pi(n)UX)_{n \in \mathbb{N}}$  is not admissible.  $\square$

The proof of Theorem 3.14 implies that  $(\pi(n)UX)_{n \in \mathbb{N}}$  is not bounded, while  $\pi(n)UX$  is topologically nilpotent for any  $n \in \mathbb{N}$ . In other word, topologically nilpotent elements converges to an almost bounded element in the strong topology. This phenomenon help us to verify that the Čech complex for Tate acyclicity is not exact. Finally we achieve the main result.

**Theorem 3.15.** *Suppose that  $|k|$  is dense in  $[0, \infty)$ . For any  $r \in (1, \infty)$ , the strongly uniform affinoid ring  $(\prod_{n \in \mathbb{N}} k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\})^{\text{ad}}$  over  $k^{\text{ad}}$  is not sheafy.*

*Proof.* Take a  $\pi \in C_{\text{bd}}(\mathbb{N}, k)$  such that  $|\pi(\mathbb{N})| \subset (\sqrt{r}, 1)$  and  $(|\pi(n)|)_{n \in \mathbb{N}} \in C_{\text{bd}}(\mathbb{N}, \mathbb{R})$  is an increasing sequence converging to 1. Put

$$\begin{aligned} \mathcal{A} &:= k\{r^{-a_i} U^i X^{a_i} \mid i \in \mathbb{N}\} \\ \mathcal{B} &:= \prod_{n \in \mathbb{N}} \mathcal{A} \\ \mathcal{X} &:= (X)_{n \in \mathbb{N}} \in \mathcal{B} \\ \mathcal{UX} &:= (UX)_{n \in \mathbb{N}} \in \mathcal{B} \\ \Pi &:= \pi = (\pi(n))_{n \in \mathbb{N}} \in C_{\text{bd}}(\mathbb{N}, k) \hookrightarrow C_{\text{bd}}(\mathbb{N}, \mathcal{A}) = \mathcal{B}. \end{aligned}$$

It suffices to verify that the Čech complex

$$0 \rightarrow \mathcal{B}\{\mathcal{X}\} \rightarrow \prod_{\sigma \in \{\pm 1\}} \mathcal{B}\{\mathcal{X}\}\{(\Pi \mathcal{UX})^\sigma\} \rightarrow \mathcal{B}\{\mathcal{X}\}\{(\Pi \mathcal{UX})^{\pm 1}\}$$

is not exact. Since  $(|\pi(n)|)_{n \in \mathbb{N}}$  is an increasing sequence, so is  $(\sup_{i \in \mathbb{N}} |\pi(n)|^i r^{b_i})_{n \in \mathbb{N}}$ . Put  $R_n := \sup_{i \in \mathbb{N}} |\pi(n)|^i r^{b_i}$  for each  $n \in \mathbb{N}$ . Note that  $(R_n)_{n \in \mathbb{N}} \in (1, \infty)^{\mathbb{N}}$  because  $(|\pi(n)|)_{n \in \mathbb{N}} \in (0, 1)^{\mathbb{N}}$  and  $(b_i)_{i \in \mathbb{N}} \in O(\log i)$ . For each  $n \in \mathbb{N}$ , denote by  $i_n \in \mathbb{N} \setminus \{0\}$  the smallest integer such that  $|\pi(n)|^{i_n} r^{b_{i_n}} = R_n$ , by  $l_n \in \mathbb{N}$  the greatest integer such that  $|\pi(n)|^{-l_n} \leq \sqrt{R_n}$ , and by  $N_n \in \mathbb{N}$  the smallest integer such that  $|\pi(n)|^{N_n} R_n \leq 1$  and  $|\pi(n)|^{N_n} r \leq 1$ . Since  $|\pi(n)|^2 \geq |\pi(0)|^2 > r$ ,  $l_n \geq 1$  for any  $n \in \mathbb{N}$ . Set

$$F := \left( \pi(n)^{l_n} \sum_{i=-N_n}^{i_n} (\pi(n)UX)^{i_n-i} T^i \right)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} (k[[U, X, T]][T^{-1}]).$$

Since  $|\pi(n)|^{-1} \sqrt{R_n} \leq |\pi(n)|^{i_n} (\pi(n)UX)^{i_n} \leq \sqrt{R_n}$  for any  $n \in \mathbb{N}$  by definition,  $F$  does not lie in the image of  $\mathcal{B}\{\mathcal{X}\}\{T, T^{-1}\}$ . We have

$$(T - \Pi \mathcal{UX})F = \left( \pi(n)^{l_n} (T^{i_n+1} - (\pi(n)UX)^{N_n+i_n+1} T^{-N_n}) \right)_{n \in \mathbb{N}}.$$

For each  $n \in \mathbb{N}$ , denote by  $e_n \in \mathcal{B}$  the image of  $1 \in \mathcal{A}$  by the zero-extension  $\mathcal{A} \hookrightarrow \mathcal{B}$  outside the  $n$ -th entry. Since  $|\pi(n)|^{l_n} \leq |\pi(n)|^{-1} \sqrt{R_n}^{-1} \xrightarrow{n \rightarrow \infty} 0$ ,  $(T - \Pi \mathcal{UX})F$  lies in the image of  $\mathcal{B}\{\mathcal{X}\}\{T, T^{-1}\}$ , and as an element of  $\mathcal{B}\{\mathcal{X}\}\{T, T^{-1}\}$ , it is the limit of the sequence

$$\left( (T - \Pi \mathcal{UX})F \sum_{l=1}^m e_l \right)_{m \in \mathbb{N}} = \left( \sum_{n=1}^m \pi(n)^{l_n} (e_n T^{i_n+1} - (\pi(n)UX)^{N_n+i_n+1} e_n T^{-N_n}) \right)_{m \in \mathbb{N}}$$

in  $\mathcal{B}\{\mathcal{X}\}\{T, T^{-1}\}$ . Each entry is an element of  $(T - \Pi\mathcal{U}\mathcal{X}) \subset \mathcal{B}\{\mathcal{X}\}\{T, T^{-1}\}$  by the computation in the proof of Theorem 3.14, and hence we have

$$(T - \Pi\mathcal{U}\mathcal{X})F \in (T - \Pi\mathcal{U}\mathcal{X})^\wedge \subset \mathcal{B}\{\mathcal{X}\}\{T, T^{-1}\}.$$

Set

$$\begin{aligned} G_+(T) &:= \sum_{n=0}^{\infty} \pi(n)^{l_n} e_n T^{i_n+1} \in \mathcal{B}\{\mathcal{X}\}\{T\} \\ G_-(T^{-1}) &:= \sum_{n=0}^{\infty} \pi(n)^{l_n} (\pi(n)UX)^{N_n+i_n+1} e_n T^{-N_n} \in \mathcal{B}\{\mathcal{X}\}\{T^{-1}\}. \end{aligned}$$

Then  $G_+ - G_- = (T - \Pi\mathcal{U}\mathcal{X})F$ , and hence  $G_+(\Pi\mathcal{U}\mathcal{X}) = G_-((\Pi\mathcal{U}\mathcal{X})^{-1})$  as elements of  $\mathcal{B}\{\mathcal{X}\}\{(\Pi\mathcal{U}\mathcal{X})^{\pm 1}\}$ . We verify that  $G_+(\Pi\mathcal{U}\mathcal{X})$  does not lie in the image of  $\mathcal{B}\{\mathcal{X}\}$  in  $\mathcal{B}\{\mathcal{X}\}\{\Pi\mathcal{U}\mathcal{X}\}$ . Let  $f = (f_n)_{n \in \mathbb{N}} \in \mathcal{B}\{\mathcal{X}\}$ , and assume that  $G_+ - f$  lies in  $(T - \Pi\mathcal{U}\mathcal{X})^\wedge \subset \mathcal{B}\{\mathcal{X}\}\{T\}$ . Set

$$F_+ := \left( \pi(n)^{l_n} \sum_{i=0}^{i_n} (\pi(n)UX)^{i_n-i} T^i \right)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} k[[U, X, T]].$$

We have

$$\begin{aligned} (T - \Pi\mathcal{U}\mathcal{X})F_+ &= \left( \pi(n)^{l_n} \left( T^{i_n+1} - (\pi(n)UX)^{i_n+1} \right) \right)_{n \in \mathbb{N}} \\ &= G_+ - \left( \pi(n)^{l_n} (\pi(n)UX)^{i_n+1} \right)_{n \in \mathbb{N}} \in \mathcal{B}\{\mathcal{X}\}\{T\}. \end{aligned}$$

Since  $G_+ - f$  lies in  $(T - \Pi\mathcal{U}\mathcal{X})^\wedge$ , there is an  $H = \sum_{h=0}^{\infty} (H_{h,n})_{n \in \mathbb{N}} T^h \in \mathcal{B}\{\mathcal{X}\}\{T\}$  such that

$$\|G_+ - f - (T - \Pi\mathcal{U}\mathcal{X})H\|_{\mathcal{B}\{T\}} < |\pi(n)|^{N_n+l_n} R_n.$$

Since  $\|\pi(n)^{l_n} (\pi(n)UX)^{i_n+1}\|_{\mathcal{A}} \geq |\pi(n)|^{l_n+1} R_n \geq |\pi(n)| \sqrt{R_n} \xrightarrow{n \rightarrow \infty} \infty$ , there is an  $n \in \mathbb{N}$  such that  $l_n > 1$  and  $\|\pi(n)^{l_n} (\pi(n)UX)^{i_n+1}\|_{\mathcal{A}\{X\}} > \max\{\|f\|_{\mathcal{B}\{X\}}, \|H\|_{\mathcal{B}\{X\}\{T\}}\}$ . Set

$$\begin{aligned} H_n &:= \sum_{h=0}^{\infty} H_{h,n} T^h \in \mathcal{A}\{X\}\{T\} \\ F_{+,n} &:= \pi(n)^{l_n} \sum_{i=0}^{i_n} (\pi(n)UX)^{i_n-i} T^i \in \mathcal{A}\{X\}[T], \end{aligned}$$

and put

$$H_{h,n} = \sum_{i=0}^{\infty} \sum_{j=b_i}^{\infty} H_{h,n,i,j} U^i X^j$$

for each  $h \in \mathbb{N}$ . We obtain

$$\begin{aligned} & \left\| \pi(n)^{l_n} (\pi(n)UX)^{i_n+1} - f_n + (T - \pi(n)UX)(F_{+,n} - H_n) \right\|_{\mathcal{A}\{X\}\{T\}} \\ &= \left\| \pi(n)^{l_n} T^{i_n+1} - f_n - (T - \pi(n)UX)H_n \right\|_{\mathcal{A}\{X\}\{T\}} \\ &\leq \|G_+ - f - (T - \Pi\mathcal{U}\mathcal{X})H\|_{\mathcal{B}\{\mathcal{X}\}\{T\}} < |\pi(n)|^{N_n+l_n} R_n. \end{aligned}$$

It implies

$$\begin{aligned} & \left\| 1 - H_{i_n,n} + \pi(n)UXH_{i_n+1,n} \right\|_{\mathcal{A}\{X\}} < |\pi(n)|^{N_n+l_n} R_n \\ & \left\| -H_{h,n} + \pi(n)UXH_{h+1,n} \right\|_{\mathcal{A}\{X\}} < |\pi(n)|^{N_n+l_n} R_n \\ & \left\| \pi(n)^{l_n} (\pi(n)UX)^{i_n+1} - f_n + (\pi(n)UX)^{i_n+1} - (\pi(n)UX)H_{0,n} \right\|_{\mathcal{A}\{X\}} < |\pi(n)|^{N_n+l_n} R_n \end{aligned}$$

for any  $h \in \mathbb{N} \setminus \{i_n\}$ . In particular, we get

$$\begin{aligned} |1 - H_{N_n,n,0,0}| &\leq |\pi(n)|^{N_n+l_n} R_n \leq |\pi(n)|^{l_n} < 1 \\ |H_{N_n,n,0,0}| &= 1. \end{aligned}$$

Considering the coefficient of  $U^i X^i$  in  $-H_{i_n-i,n} + \pi(n)UXH_{i_n-i+1,n}$  inductively on  $i \in \mathbb{N} \cap [1, i_n]$ , we acquire

$$\begin{aligned} & \left\| (\pi(n)UX)^{i_n} - (UX)^{i_n} H_{0,n,i_n,i_n} \right\|_{\mathcal{A}\{X\}} < |\pi(n)|^{N_n+l_n} R_n < R_n = \left\| (\pi(n)UX)^{i_n} \right\|_{\mathcal{A}\{X\}} \\ & \left\| (UX)^{i_n} H_{0,n,i_n,i_n} \right\|_{\mathcal{A}\{X\}} = \left\| (\pi(n)UX)^{i_n} \right\|_{\mathcal{A}\{X\}} = R_n. \end{aligned}$$

Therefore we gain

$$\begin{aligned} & \left\| (\pi(n)UX)^{i_n+1} - (\pi(n)UX)H_{0,n,i_n,i_n} \right\|_{\mathcal{A}\{X\}} \\ &\leq \|\pi(n)UX\|_{\mathcal{A}\{X\}} \left\| (\pi(n)UX)^{i_n} - (UX)^{i_n} H_{0,n,i_n,i_n} \right\|_{\mathcal{A}\{X\}} < |\pi(n)|^{N_n+l_n+1} rR_n \\ &\leq |\pi(n)|^{l_n+1} R_n = |\pi(n)|^{l_n+1} \left\| (\pi(n)UX)^{i_n} \right\|_{\mathcal{A}\{X\}} \leq \left\| \pi(n)^{l_n} (\pi(n)UX)^{i_n+1} \right\|_{\mathcal{A}\{X\}} \end{aligned}$$

and hence

$$\begin{aligned} & \left\| \pi(n)^{l_n} (\pi(n)UX)^{i_n+1} - f_n + (\pi(n)UX)^{i_n+1} - (\pi(n)UX)H_{0,n} \right\|_{\mathcal{A}\{X\}} \\ &\geq \left\| \pi(n)^{l_n} (\pi(n)UX)^{i_n+1} \right\|_{\mathcal{A}\{X\}} \geq |\pi(n)|^{l_n+1} R_n \end{aligned}$$

because  $\|f_n\|_{\mathcal{A}\{X\}} \leq \|f\|_{\mathcal{B}} < \left\| \pi(n)^{l_n} (\pi(n)UX)^{i_n+1} \right\|_{\mathcal{A}\{X\}}$ . It contradicts the inequality

$$\left\| \pi(n)^{l_n} (\pi(n)UX)^{i_n+1} - f_n + (\pi(n)UX)^{i_n+1} - (\pi(n)UX)H_{0,n} \right\|_{\mathcal{A}\{X\}} < |\pi(n)|^{N_n+l_n} R_n.$$

Thus  $G_+(\Pi\mathcal{U}\mathcal{X})$  does not lie in the image of  $\mathcal{B}\{\mathcal{X}\}$ . We conclude that the sequence

$$0 \rightarrow \mathcal{B}\{\mathcal{X}\} \rightarrow \prod_{\sigma \in \{\pm 1\}} \mathcal{B}\{\mathcal{X}\} \{(\Pi\mathcal{U}\mathcal{X})^\sigma\} \rightarrow \mathcal{B}\{\mathcal{X}\} \{(\Pi\mathcal{U}\mathcal{X})^{\pm 1}\}$$

is not exact, and  $\mathcal{B}^{\text{ad}}$  is not sheafy. □

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